# Portfolios that Contain Risky Assets

## Portfolio Models 5.

## Leveraged Portfolios

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## Portfolios that Contain Risky Assets
## Part I: Portfolio Models

# Portfolio Models 5. Leveraged Portfolios

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## Leveraged Portfolios

**Introduction.** Leveraged portfolios are ones that take short positions. Short positions can offer the promise of great reward, but come with the potential for greater losses. They are favored by quantitative hedge funds, notable examples of which are the Medallion, RIEF, and RIDA funds run by Renaissance Technologies. They were also favored by major investment banks during the first decade of the $21^{\text{st}}$ century, and played a major role in bringing about the subsequent great recession. They had a similar role in bringing about the great depression seventy eight years earlier. In fact, they have played a role in every major market bubble crash, such as the dot-com crash of 2000.

Because leveraged portfolios can lead to systemic risk, they are something about which every investor should have some understanding. We will try to gain such an understanding by building a simple model of leveraged Markowitz portfolios.

**Limited Leverage Portfolios.** The class $\Omega$ of solvent Markowitz portfolios is unrealistic because it allows an investor to take short positions without much collateral. In practice such positions are restricted by *credit limits*.

If we assume that in each case the lender is the broker and the collateral is part of the portfolio then a simple model for credit limits is to constrain the total short position of the portfolio to be at most a positive multiple $\ell$ of the portfolio value. The value of $\ell$ is called the *leverage limit* of the portfolio and will depend upon market conditions, but brokers will often allow $\ell > 1$ and seldom allow $\ell > 5$.

Just because a broker allows a particular value of $\ell$ does not mean it is in the best interest of an investor to build a portfolio with that value of $\ell$. We will use this model to understand what values of $\ell$ might not be prudent. This understanding will give us a measure of when markets are stressed.

In order to derive constraints on the allocations based upon this simple model, we decompose any $\mathbf{f}$ into its long and short positions as

$$\mathbf{f} = \mathbf{f}_+ - \mathbf{f}_- \,, \tag{1}$$

where $f_i^{\pm}$, the $i^{\text{th}}$ entry of $\mathbf{f}_{\pm}$, is given by

$$f_i^+ = \max\{f_i \,, 0\} \,, \qquad f_i^- = \max\{-f_i \,, 0\} \,.$$

This is the so-called *long-short decomposition* of $\mathbf{f}$. The vectors $\mathbf{f}_+$ and $\mathbf{f}_-$ in this decomposition are characterized by

$$\mathbf{f}_+ \geq \mathbf{0} \,, \qquad \mathbf{f}_- \geq \mathbf{0} \,, \qquad \mathbf{f}_+^{\mathsf{T}} \mathbf{f}_- = 0 \,.$$

The constraint that the multiple of the portfolio value held in short positions is bounded by a leverage limit $\ell$ can be expressed as

$$\mathbf{1}^{\mathsf{T}} \mathbf{f}_- \leq \ell \,. \tag{2}$$

It follows from the constraint $\mathbf{1}^\mathsf{T}\mathbf{f} = 1$ and decompostion (1) that

$$1 = \mathbf{1}^\mathsf{T}\mathbf{f} = \mathbf{1}^\mathsf{T}\mathbf{f}_+ - \mathbf{1}^\mathsf{T}\mathbf{f}_- \,.$$

We also have

$$|\mathbf{f}| = \mathbf{1}^\mathsf{T}\mathbf{f}_+ + \mathbf{1}^\mathsf{T}\mathbf{f}_- \,,$$

where $|\mathbf{f}|$ denotes the $\ell^1$-norm of $\mathbf{f}$, which is defined by

$$|\mathbf{f}| = \sum_{i=1}^{N} |f_i| \,.$$

Notice that $1 = |\mathbf{1}^\mathsf{T}\mathbf{f}| \leq |\mathbf{f}|$. By first adding and subtracting the top relation above from the second, and then dividing by $2$, we obtain

$$\mathbf{1}^\mathsf{T}\mathbf{f}_+ = \frac{|\mathbf{f}| + 1}{2} \,, \qquad \mathbf{1}^\mathsf{T}\mathbf{f}_- = \frac{|\mathbf{f}| - 1}{2} \,. \tag{3}$$

These are the multiples of the portfolio value that are held in long and short positions respectively. Notice that $\mathbf{1}^\mathsf{T}\mathbf{f}_+ \geq 1$ and that $\mathbf{1}^\mathsf{T}\mathbf{f}_- \geq 0$.

Constraint (2) that bounds the multiple of the portfolio value held in short positions by $\ell$ thereby becomes

$$\frac{|\mathbf{f}| - 1}{2} = \mathbf{1}^{\mathsf{T}}\mathbf{f}_{-} \leq \ell \,.$$

We thereby see that if $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$ then

$$\mathbf{1}^{\mathsf{T}}\mathbf{f}_{-} \leq \ell \qquad \Longleftrightarrow \qquad |\mathbf{f}| \leq 1 + 2\ell \,.$$

Therefore the set of allocations for Markowitz portfolios with a leverage limit $\ell \in [0, \infty)$ is

$$\Pi_{\ell} = \left\{ \mathbf{f} \in \mathbb{R}^{N} \;:\; \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1 \,,\; |\mathbf{f}| \leq 1 + 2\ell \right\}. \tag{4}$$

It is clear that if $\ell, \ell' \in [0, \infty)$ then

$$\ell \leq \ell' \qquad \Longrightarrow \qquad \Pi_{\ell} \subset \Pi_{\ell'} \,.$$

Recall that the set of allocations for long Markowitz portfolios is

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\, \mathbf{f} \geq \mathbf{0} \right\}. \tag{5}$$

We now show that *the limited leverage Markowitz portfolios with leverage limit $\ell = 0$ are exactly the long Markowitz portfolios*.

**Fact 1.** We have $\Pi_0 = \Lambda$.

**Proof.** Let $\mathbf{f} \in \Pi_0$. Let $\mathbf{f} = \mathbf{f}_+ - \mathbf{f}_-$ be the long-short decomposition of $\mathbf{f}$ given by (1). Because $\mathbf{f}_- \geq \mathbf{0}$ while $\mathbf{1}^\mathsf{T}\mathbf{f}_- \leq \ell = 0$, we conclude that $\mathbf{f}_- = \mathbf{0}$. Therefore $\mathbf{f} \in \Lambda$.

Conversely, if $\mathbf{f} \in \Lambda$ then $\mathbf{f}_- = \mathbf{0}$, so $\mathbf{1}^\mathsf{T}\mathbf{f}_- = 0$, whereby $\mathbf{f} \in \Pi_0$. $\square$

**Unlimited Leverage Portfolios.** If we take the union of the sets $\Pi_\ell$ over $\ell \in [0, \infty)$ then we obtain the set

$$\Pi_\infty = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1 \right\}. \tag{6}$$

It is clear from (4) that if $\ell \in [0, \infty)$ then $\Pi_\ell \subset \Pi_\infty$.

The set $\Pi_\infty$ is the set of allocations for all Markowitz portfolios. It contains the set $\Omega$ of allocations for solvent Markowitz portfolios. It is unrealistic because it allows investors to take short positions with almost no collateral. However, it has the virtue that its only constraint is $\mathbf{1}^\mathsf{T}\mathbf{f} = 1$, which is an equality constraint. This fact makes it easier to use in many settings than the sets $\Pi_\ell$ with $\ell \in [0, \infty)$, which involve inequality constraints. By using $\Pi_\infty$ we are often able to derive analytical expressions that offer insight. This will be illustrated in the next lecture.

**Solvent Leveraged Portfolios.** Recall that for a given price ratio history $\{\boldsymbol{\rho}(d)\}_{d=1}^{D}$ the set of allocations for solvent Markowitz portfolios is

$$\Omega = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1 \,,\, 0 < \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} \;\; \forall d \right\} , \tag{7a}$$

the set of allocations for Markowitz portfolios with value ratios bounded below by $\rho \in (0, \infty)$ is

$$\Omega_\rho = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1 \,,\, \rho \leq \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} \;\; \forall d \right\} , \tag{7b}$$

and the set of allocations for Markowitz portfolios with value ratios that are contained within $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$ is

$$\Omega_{[\underline{\rho},\overline{\rho}]} = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1 \,,\, \underline{\rho} \leq \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} \leq \overline{\rho} \;\; \forall d \right\} . \tag{7c}$$

Here we will give bounds on the leverage limit $\ell$ that will characterize when $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$, when $\Pi_\ell \subset \Omega_\rho$, and when $\Pi_\ell \subset \Omega$.

These bounds will be express in terms of the quantities $\rho_{\mathsf{mn}}(d)$ and $\rho_{\mathsf{mx}}(d)$ defined by

$$
\begin{aligned}
\rho_{\mathsf{mn}}(d) &= \min\left\{\rho_i(d) \,:\, i = 1, \cdots, N\right\}, \\
\rho_{\mathsf{mx}}(d) &= \max\left\{\rho_i(d) \,:\, i = 1, \cdots, N\right\}.
\end{aligned}
\tag{8}
$$

These are the price ratios of the worst and the best performing asset on trading day $d$. We expect that $0 < \rho_{\mathsf{mn}}(d) < \rho_{\mathsf{mx}}(d)$ on every trading day.

**Remark 1.** On most trading days a large, well-balanced portfolio will have an asset that decreases in value and another asset that increases in value. For such days we will have

$$
0 < \rho_{\mathsf{mn}}(d) < 1 < \rho_{\mathsf{mx}}(d) .
$$

For small portfolios it is not uncommon for $0 < \rho_{\mathsf{mn}}(d) < \rho_{\mathsf{mx}}(d) < 1$ on days when the whole market goes down, or for $1 < \rho_{\mathsf{mn}}(d) < \rho_{\mathsf{mx}}(d)$ on days when the whole market goes up.

**Remark 2.** Recall from the last lecture that $\Lambda \subset \Omega_{[\rho_{\mathsf{mn}},\rho_{\mathsf{mx}}]}$ where

$$\rho_{\mathsf{mn}} = \min_{d} \Big\{ \rho_{\mathsf{mn}}(d) \Big\} \,, \qquad \rho_{\mathsf{mx}} = \max_{d} \Big\{ \rho_{\mathsf{mx}}(d) \Big\} \,, \tag{9}$$

and that $\Omega_{[\rho_{\mathsf{mn}},\rho_{\mathsf{mx}}]}$ is the smallest such set containing $\Lambda$. From **Fact 1** and definitions (4) and (6) we see that $\Lambda = \Pi_0 \subset \Pi_\ell$ for every $\ell \in [0,\infty]$, whereby we conclude that:

- a necessary condition for $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$ is $[\underline{\rho},\overline{\rho}] \supset [\rho_{\mathsf{mn}},\rho_{\mathsf{mx}}]$;

- a necessary condition for $\Pi_\ell \subset \Omega_\rho$ is $\rho \leq \rho_{\mathsf{mn}}$.

We are now ready to state our characterizations.

**Fact 2.** Let $\ell \in [0,\infty)$ and $[\underline{\rho},\overline{\rho}] \subset (0,\infty)$. Then $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$ if and only if $[\rho_{\text{mn}},\rho_{\text{mx}}] \subset [\underline{\rho},\overline{\rho}]$ and

$$\ell \le \min_d \left\{ \frac{\rho_{\text{mn}}(d) - \underline{\rho}}{\rho_{\text{mx}}(d) - \rho_{\text{mn}}(d)}, \frac{\overline{\rho} - \rho_{\text{mx}}(d)}{\rho_{\text{mx}}(d) - \rho_{\text{mn}}(d)} \right\} . \tag{10a}$$

**Fact 3.** Let $\ell \in [0,\infty)$ and $\rho \in (0,\infty)$. Then $\Pi_\ell \subset \Omega_\rho$ if and only if $\rho \le \rho_{\text{mn}}$ and

$$\ell \le \min_d \left\{ \frac{\rho_{\text{mn}}(d) - \rho}{\rho_{\text{mx}}(d) - \rho_{\text{mn}}(d)} \right\} . \tag{10b}$$

**Fact 4.** Let $\ell \in [0,\infty)$. Then $\Pi_\ell \subset \Omega$ if and only if

$$\ell < \min_d \left\{ \frac{\rho_{\text{mn}}(d)}{\rho_{\text{mx}}(d) - \rho_{\text{mn}}(d)} \right\} . \tag{10c}$$

**Proofs.** We see from the definitions of $\rho_{\mathsf{mn}}(d)$ and $\rho_{\mathsf{mx}}(d)$ given in (8) that $\boldsymbol{\rho}(d)$ satisfies the entrywise inequalities

$$\rho_{\mathsf{mn}}(d)\,\mathbf{1} \leq \boldsymbol{\rho}(d) \leq \rho_{\mathsf{mx}}(d)\,\mathbf{1}\,.$$

These inequalities will be equalities for those entries corresponding to the worst and best performing assets respectively.

Let $\mathbf{f} = \mathbf{f}_{+} - \mathbf{f}_{-}$ be the long-short decomposition of $\mathbf{f}$ given by (1). Because $\mathbf{f}_{\pm} \geq \mathbf{0}$, the above entrywise inequalities yield the bounds

$$\rho_{\mathsf{mn}}(d)\,\mathbf{1}^{\mathsf{T}}\mathbf{f}_{\pm} \leq \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}_{\pm} \leq \rho_{\mathsf{mx}}(d)\,\mathbf{1}^{\mathsf{T}}\mathbf{f}_{\pm}\,. \tag{11}$$

These inequalities will be equalities when the only nonneutral positions are held in the worst and best performing assets respectively.

We see from the bounds (11), the formulas (3) for $\mathbf{1}^{\mathsf{T}}\mathbf{f}_{\pm}$, and definition (4) of $\Pi_\ell$ that for every $\mathbf{f} \in \Pi_\ell$ a lower bound for $\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}$ is

$$\begin{aligned}
\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} &= \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}_{+} - \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}_{-} \\
&\geq \rho_{\mathsf{mn}}(d)\, \mathbf{1}^{\mathsf{T}}\mathbf{f}_{+} - \rho_{\mathsf{mx}}(d)\, \mathbf{1}^{\mathsf{T}}\mathbf{f}_{-} \\
&= \rho_{\mathsf{mn}}(d)\, \frac{|\mathbf{f}| + 1}{2} - \rho_{\mathsf{mx}}(d)\, \frac{|\mathbf{f}| - 1}{2} \\
&= \frac{\rho_{\mathsf{mx}}(d) + \rho_{\mathsf{mn}}(d)}{2} - \frac{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}{2}\, |\mathbf{f}| \\
&\geq \frac{\rho_{\mathsf{mx}}(d) + \rho_{\mathsf{mn}}(d)}{2} - \frac{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}{2}\, (1 + 2\ell) \\
&= \rho_{\mathsf{mn}}(d) - \Big(\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)\Big)\, \ell\,.
\end{aligned}$$

This lower bound will be greater than or equal to $\underline{\rho}$ if and only if

$$\ell \leq \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}\,. \tag{12a}$$

We see from the bounds (11), the formulas (3) for $\mathbf{1}^\mathsf{T}\mathbf{f}_\pm$, and definition (4) of $\Pi_\ell$ that for every $\mathbf{f} \in \Pi_\ell$ an upper bound for $\boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}$ is

$$\begin{aligned}
\boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} &= \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}_+ - \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}_- \\
&\leq \rho_{\mathsf{mx}}(d)\mathbf{1}^\mathsf{T}\mathbf{f}_+ - \rho_{\mathsf{mn}}(d)\mathbf{1}^\mathsf{T}\mathbf{f}_- \\
&= \rho_{\mathsf{mx}}(d)\,\frac{|\mathbf{f}|+1}{2} - \rho_{\mathsf{mn}}(d)\,\frac{|\mathbf{f}|-1}{2} \\
&= \frac{\rho_{\mathsf{mx}}(d)-\rho_{\mathsf{mn}}(d)}{2}\,|\mathbf{f}| - \frac{\rho_{\mathsf{mx}}(d)+\rho_{\mathsf{mn}}(d)}{2} \\
&\leq \frac{\rho_{\mathsf{mx}}(d)-\rho_{\mathsf{mn}}(d)}{2}\,(1+2\ell) - \frac{\rho_{\mathsf{mx}}(d)+\rho_{\mathsf{mn}}(d)}{2} \\
&= \rho_{\mathsf{mx}}(d) + \Big(\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)\Big)\,\ell\,.
\end{aligned}$$

This upper bound will be less than or equal to $\overline{\rho}$ if and only if

$$\ell \leq \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}\,. \tag{12b}$$

First assume that $[\underline{\rho}, \overline{\rho}] \supset [\rho_{\mathsf{mn}}, \rho_{\mathsf{mx}}]$ and that $\ell$ satisfies bound (10a). Then $\ell$ satisfies the bounds (12a) and (12b) for every $d = 1, \cdots, D$. Therefore $\Pi_\ell \subset \Omega_{[\underline{\rho}, \overline{\rho}]}$.

Now assume that $\Pi_\ell \subset \Omega_{[\underline{\rho}, \overline{\rho}]}$. **Remark 2** shows that $[\underline{\rho}, \overline{\rho}] \supset [\rho_{\mathsf{mn}}, \rho_{\mathsf{mx}}]$. If $\ell$ does not satisfy bound (10a) then for some $d$ either

$$\ell > \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \qquad \text{or} \qquad \ell > \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)},$$

then we can construct an $\mathbf{f} \in \Pi_\ell$ such that either $\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} < \underline{\rho}$ in the first case by being short in a best performing asset and long in a worst performing asset, or $\overline{\rho} < \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}$ in the second case by being long in a best performing asset and short in a worst performing asset.

Therefore we have proved **Fact 2**.

Next assume that $\rho \leq \rho_{\text{mn}}$ and $\ell$ satisfies bound (10b). Then $\ell$ satisfies the bound (12a) for every $d = 1, \cdots, D$ with $\underline{\rho} = \rho$. Therefore $\Pi_\ell \subset \Omega_\rho$.

Now assume that $\Pi_\ell \subset \Omega_\rho$. **Remark 2** shows that $\rho \leq \rho_{\text{mn}}$. If $\ell$ does not satisfy bound (10b) then for some $d$

$$\ell > \frac{\rho_{\text{mn}}(d) - \rho}{\rho_{\text{mx}}(d) - \rho_{\text{mn}}(d)},$$

then we can construct an $\mathbf{f} \in \Pi_\ell$ such that $\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} < \rho$ by being short in a best performing asset and long in a worst performing asset.

Therefore we have proved **Fact 3**.

Finally, because $\Omega$ is the union of all the $\Omega_\rho$, it follows from **Fact 3** that $\Pi_\ell \subset \Omega$ for some $\ell \geq 0$ if and only if $\ell$ satisfy bound (10c).

Therefore we have proved **Fact 4**. $\square$

**Leverage Limit Bound.** We can restate **Fact 4** as *every portfolio in $\Pi_\ell$ is solvent if and only if $\ell \in [0, \ell_{\mathsf{sol}})$*, where by (10c) the leverage limit upper bound $\ell_{\mathsf{sol}}$ is

$$\ell_{\mathsf{sol}} = \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} = \frac{1}{\max_d \left\{ \frac{\rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mn}}(d)} \right\} - 1} . \tag{13}$$

It depends upon the ratios $\rho_{\mathsf{mx}}(d)/\rho_{\mathsf{mn}}(d)$ over the history considered. These ratios can be close to 1 on days when the entire market moves up or down by a substantial amount. They can be largest on days when the market does not make a major move.

One use of this bound is to monitor stress in the market. The lower $\ell_{\mathsf{sol}}$, the more stress the market is under. Another use is to determine a safe leverage limit for your own portfolio. It is wise to consider a long history when computing the bound for this purpose.

When $\ell \in [0, \ell_{\mathsf{sol}})$ we can identify an interval $[\underline{\rho}, \overline{\rho}]$ such that $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$. This fact can be used to select $\ell$ so that $\Pi_\ell$ falls within a target $\Omega_{[\underline{\rho},\overline{\rho}]}$.

**Fact 5.** If $\ell \in [0, \ell_{\mathsf{sol}})$ then $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$ where

$$\underline{\rho} = \left(1 - \frac{\ell}{\ell_{\mathsf{sol}}}\right) \rho_{\mathsf{mn}} \,, \qquad \overline{\rho} = \left(1 + \frac{\ell}{1 + \ell_{\mathsf{sol}}}\right) \rho_{\mathsf{mx}} \,. \tag{14}$$

Moreover, because $\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega_{\underline{\rho}}$, we have $\Pi_\ell \subset \Omega_{\underline{\rho}}$.

**Proof.** Let $\ell \in [0, \ell_{\mathsf{sol}})$. Let $\underline{\rho}$ and $\overline{\rho}$ be given by (14). Then

$$\begin{aligned} \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} &\geq \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d) - \left(1 - \frac{\ell}{\ell_{\mathsf{sol}}}\right) \rho_{\mathsf{mn}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} \\ &= \frac{\ell}{\ell_{\mathsf{sol}}} \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} = \ell \,. \end{aligned}$$

Similarly,

$$\min_d \left\{ \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} \geq \min_d \left\{ \frac{\left(1 + \frac{\ell}{1+\ell_{\mathsf{sol}}}\right) \rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\}$$

$$= \frac{\ell}{1 + \ell_{\mathsf{sol}}} \min_d \left\{ \frac{\rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\}$$

$$= \frac{\ell}{1 + \ell_{\mathsf{sol}}} \left(1 + \ell_{\mathsf{sol}}\right) = \ell \,.$$

Because $[\underline{\rho}, \overline{\rho}] \supset [\rho_{\mathsf{mn}}, \rho_{\mathsf{mx}}]$ and because

$$\min_d \left\{ \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \,, \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} \geq \ell \,,$$

we conclude by **Fact 2** that $\Pi_\ell \subset \Omega_{[\underline{\rho}, \overline{\rho}]}$. □

**Remark.** Generally there is an interval $[\underline{\rho}, \overline{\rho}]$ such that $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$ that is smaller than the one given by (14). However, if $\rho_{\mathrm{mn}}(d)$ is close to $\rho_{\mathrm{mn}}$ and $\rho_{\mathrm{mx}}(d)$ is close to $\rho_{\mathrm{mx}}$ on days when $\rho_{\mathrm{mx}}(d)/\rho_{\mathrm{mn}}(d)$ is close to its maximum then the values of $\underline{\rho}$ and $\overline{\rho}$ given by (14) will be near optimal.

**Remark.** It is natural to ask why an investor who maintains a long portfolio should care about bounds on leverage limits. The answer is that bounds on leverage limits can fall well before a market bubble collapses. During a bubble some investors will succumb to the temptation of taking highly leveraged positions. The most highly leveraged investors will be stressed when bounds on leverage limits fall. They may have to shed some of their position to cover their margins. This creates market volatility, which in turn can drive bounds on leverage limits down further. This can go on for quite a while before the market turns down — if it turns down. Observant long investors can use this time move into a more conservative position. It is wise to use short histories when computing these bounds for this purpose.