Lecture 1: Graphs, Adjacency Matrices, Graph Laplacian

Radu Balan

January 31, 2017

Definitions

$G = (\mathcal{V}, \mathcal{E})$

An *undirected graph* G is given by two pieces of information: a set of *vertices* $\mathcal{V}$ and a set of *edges* $\mathcal{E}$, $G = (\mathcal{V}, \mathcal{E})$.

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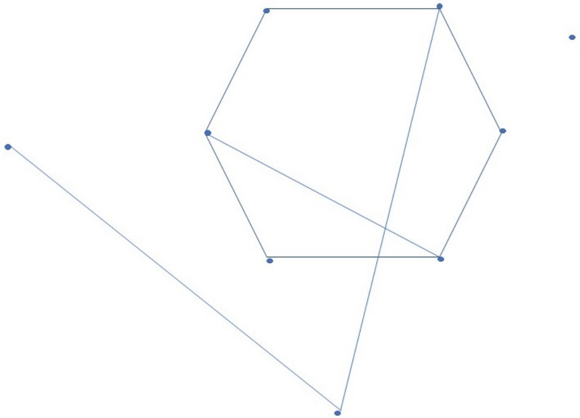

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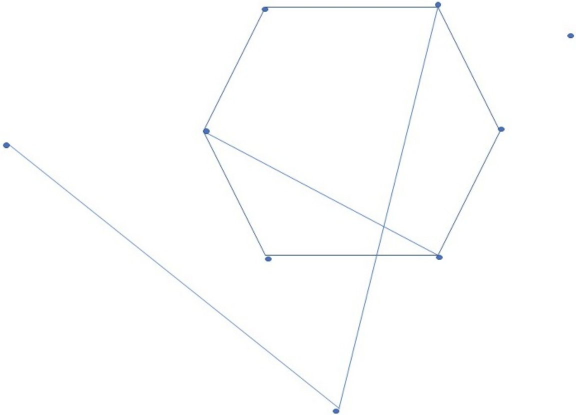

$\mathcal{V} = ?$
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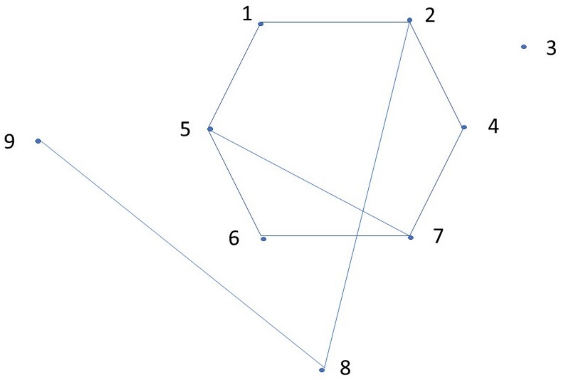

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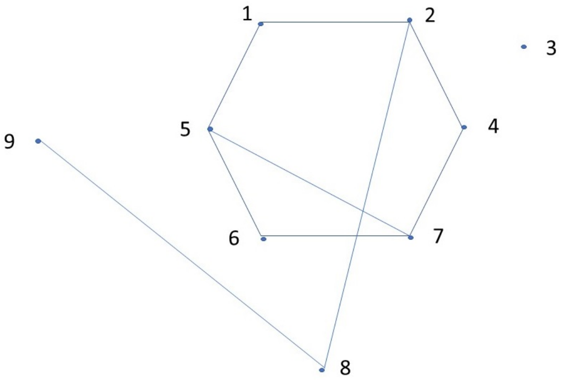

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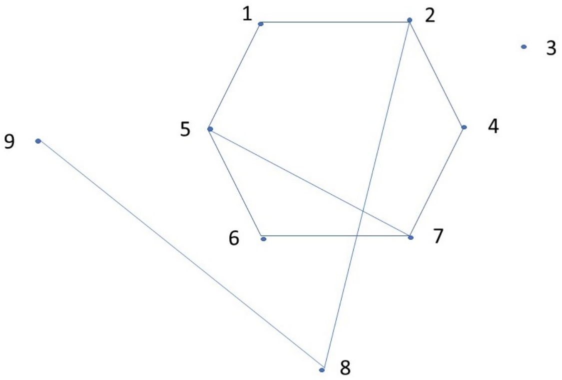

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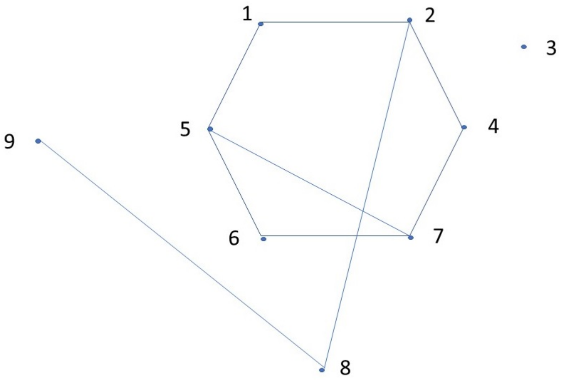

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$$n = 9 \ \ vertices$$

$$m = 9 \ \ edges$$

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- Weighted Graphs: Each edge has an associated weight. A weighted graph is defined by a triple $(\mathcal{V}, \mathcal{E}, w)$, where $w : \mathcal{E} \to \mathbb{R}$ is a weight function.

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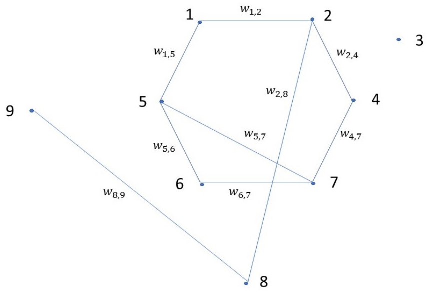

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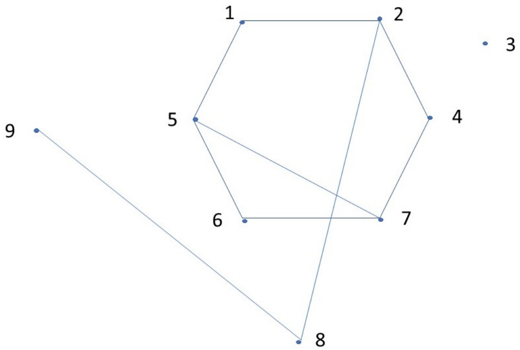
Paths

Concept: A path is a sequence of edges where the right vertex of one edge coincides with the left vertex of the following edge.

Example:



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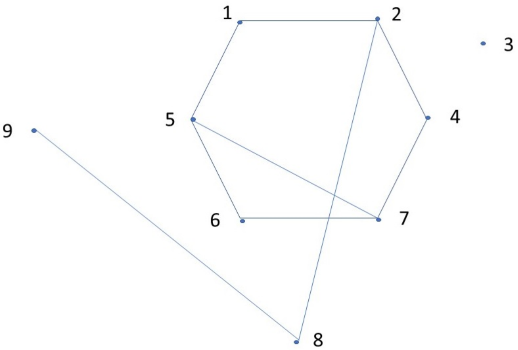

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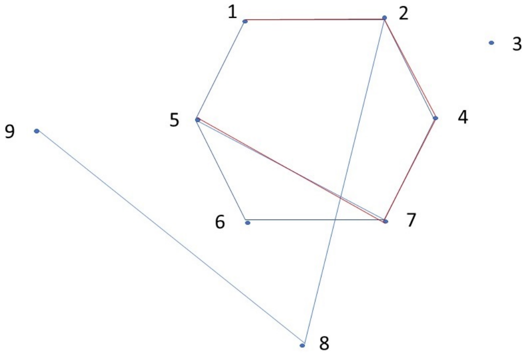

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Graph Attributes (Properties):

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A complete graph with n vertices has $m = \begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n-1)}{2}$ edges.

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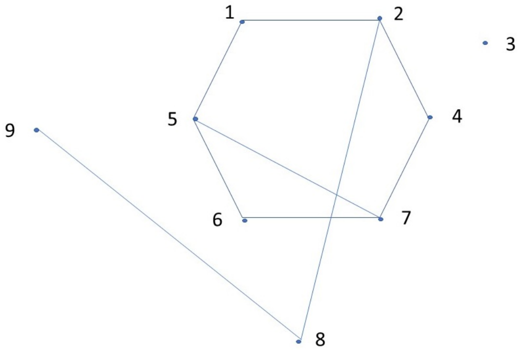

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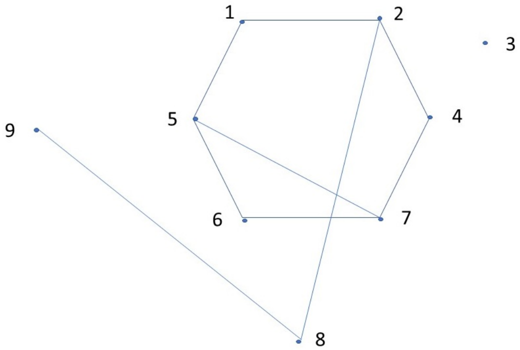

- This graph is not connected.
- It is not complete.
- It is the union of two connected graphs.

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In a connected graph the distance between any two vertices is finite.
In a complete graph the distance between any two distinct vertices is 1.
The converses are also true:

1. If $\forall x, y \in \mathcal{E}$, $d(x, y) < \infty$ then $(\mathcal{V}, \mathcal{E})$ is connected.
2. If $\forall x \neq y \in \mathcal{E}$, $d(x, y) = 1$ then $(\mathcal{V}, \mathcal{E})$ is complete.

Definitions
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Graph diameter: The diameter of a graph $G = (\mathcal{V}, \mathcal{E})$ is the largest distance between two vertices of the graph:

$$D(G) = \max_{x,y \in \mathcal{V}} d(x,y)$$

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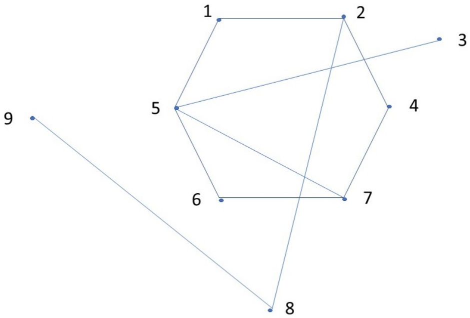

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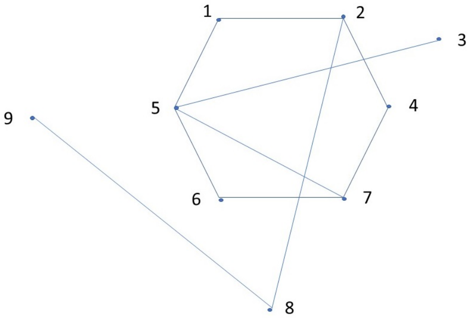

$$D = 5 = d(6,9) = d(3,9)$$

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The Adjacency Matrix

For a graph $G = (\mathcal{V}, \mathcal{E})$ the adjacency matrix is the $n \times n$ matrix A defined by:

$$A_{i,j} = \begin{cases} 1 & if \quad (i,j) \in \mathcal{E} \\ 0 & \quad otherwise \end{cases}$$

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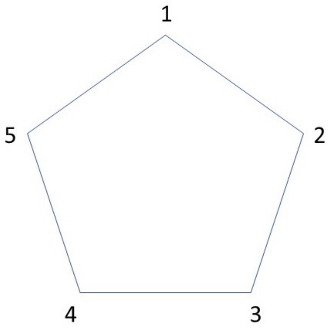

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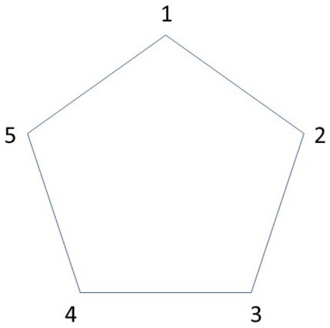

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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The Adjacency Matrix

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$$A^T = A$$

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For directed graphs the adjacency matrix may not be symmetric.
For weighted graphs $G = (\mathcal{V}, \mathcal{E}, W)$, the weight matrix W is simply given by

$$W_{i,j} = \begin{cases} w_{i,j} & \text{if} \quad (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Vertex Degree
$d(v)$

For an undirected graph $G = (\mathcal{V}, \mathcal{E})$, let $d(v)$ denote the number of edges at vertex $v \in \mathcal{V}$. The number $d(v)$ is called the degree (or valency) of vertex v.

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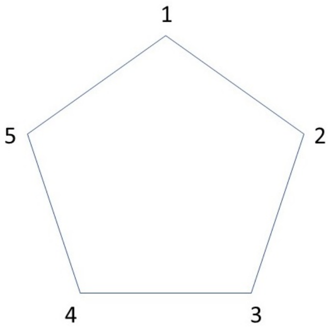

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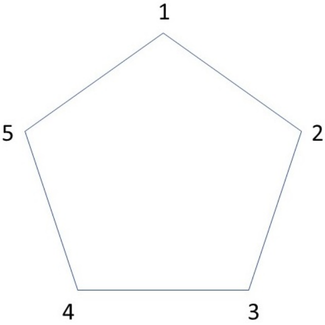

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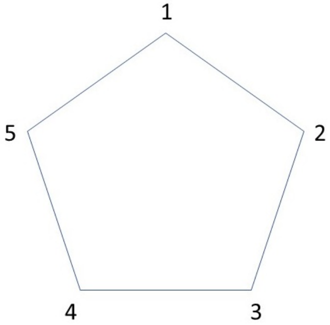

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Note: $d(i) = \sum_{j=1}^{5} A_{i,j}$

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Matrix D

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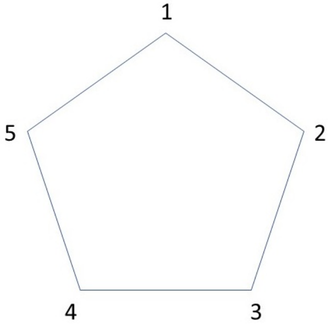

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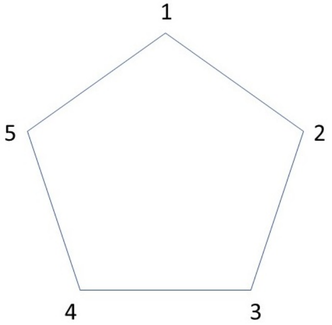

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Graph Laplacian
Δ

For a graph $G = (\mathcal{V}, \mathcal{E})$ the graph Laplacian is the $n \times n$ symmetric matrix Δ defined by:

$$\Delta = D - A$$

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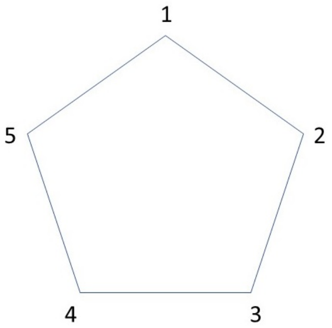

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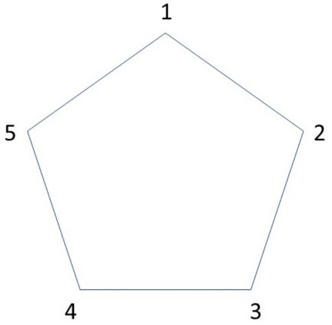

$$\Delta = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Graph Laplacian
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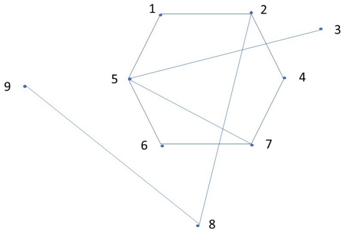

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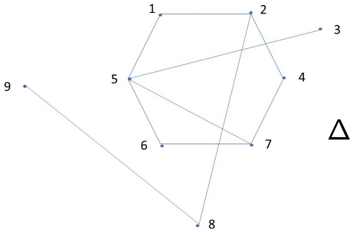

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Normalized Laplacians
$\tilde{\Delta}$

Normalized Laplacian: (using pseudo-inverses)

$$\tilde{\Delta} = D^{-1/2}\Delta D^{-1/2} = I - D^{-1/2}AD^{-1/2}$$

$$\tilde{\Delta}_{i,j} = \begin{cases} 1 & if \quad i = j \text{ and } d_i > 0 \\ -\frac{1}{\sqrt{d(i)d(j)}} & if \quad (i,j) \in \mathcal{E} \\ 0 & otherwise \end{cases}$$

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Note:

$$\Delta D^{-1} = I - AD^{-1} = L^T \quad ; \quad (D^{-1})_{kk} = (D^{-1/2})_{kk} = 0 \text{ if } d(k) = 0$$

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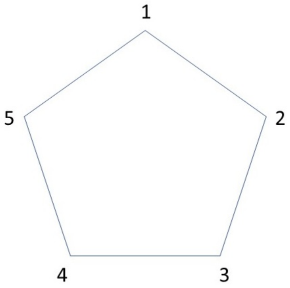

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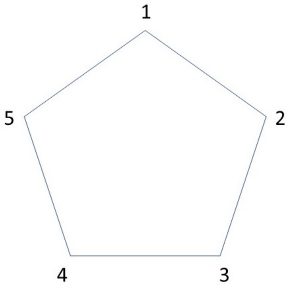

$$\tilde{\Delta} = \begin{bmatrix} 1 & -0.5 & 0 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 & 0 \\ 0 & -0.5 & 1 & -0.5 & 0 \\ 0 & 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & 0 & -0.5 & 1 \end{bmatrix}$$

Spectral Analysis
Eigenvalues and Eigenvectors

Recall the eigenvalues of a matrix T are the zeros of the characteristic polynomial:

$$p_T(z) = det(zI - T) = 0.$$

There are exactly n eigenvalues (including multiplicities) for a $n \times n$ matrix T. The set of eigenvalues is calles its *spectrum*.

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Remark. Since $det(A_1 A_2) = det(A_1)det(A_2)$ and $L = D^{-1/2}\tilde{\Delta}D^{1/2}$ it follows that $eigs(\tilde{\Delta}) = eigs(L) = eigs(L^T)$.

Spectral Analysis
Rayleigh Quotient

Recall the following result:

Theorem

Assume T is a real symmetric $n \times n$ matrix. Then:

1. *All eigenvalues of T are real numbers.*
2. *There are n eigenvectors that can be normalized to form an orthonormal basis for $\mathbb{R}^n$.*
3. *The largest eigenvalue λ_{max} and the smallest eigenvalue λ_{min} satisfy*

$$\lambda_{max} = \max_{x \neq 0} \frac{\langle Tx, x\rangle}{\langle x, x\rangle} \quad , \quad \lambda_{min} = \min_{x \neq 0} \frac{\langle Tx, x\rangle}{\langle x, x\rangle}$$

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Rayleigh Quotient

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In particular we say T is positive semidefinite $T \geq 0$ if $\langle Tx, x \rangle \geq 0$ for every x.
It follows that T is positive semidefinite if and only if every eigenvalue of T is positive semidefinite (i.e. non-negative).

References

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