Portfolios that Contain Risky Assets

Portfolio Models 7.

Survey of Markowitz Portfolio Models

C. David Levermore
University of Maryland, College Park

Math 420: *Mathematical Modeling*
March 11, 2016 version

Portfolio Models 1. Risk and Reward
2. Markowitz Portfolios
3. Basic Markowitz Portfolio Theory
4. Portfolios with Risk-Free Assets
5. Long Portfolios
6. Long Portfolios with a Safe Investment

Stochastic Models 1. Independent, Identically-Distributed Models
2. Law of Large Numbers (Kelly) Objectives
3. Growth Rate Estimators
4. Central Limit Theorem Objectives
5. Optimization of Mean-Variance Objectives
6. Utility Function Objectives

Survey of Markowitz Portfolio Models

1. Introduction
2. Long Portfolios
3. Leveraged Portfolios
4. Unrestricted Portfolios
5. Price Ratios of Assets
6. Value Ratios of Markowitz Portfolios
7. Solvent Portfolios
8. Bounded Value-Ratio Portfolios
9. Bounded Below Value-Ratio Portfolios
10. Relationships with Leveraged Portfolios

Survey of Markowitz Portfolio Models

Introduction. So far we have considered Markowitz portfolios that are either long, leveraged, unrestricted, or solvent. Here we will review these portfolio models and introduce a few more. The new models will give some insight into how leverage limits can be chosen for leveraged portfolios. In other words, they can help manage risk.

The new portfolio models will be built upon *value ratios* for portfolios, which are built from *price ratios* for individual assets. These notions were used to construct solvent portfolios. We will review both of these notions and the construction of solvent portfolios before introducing the new models.

Long Portfolios. For long Markowitz portfolios with no risk-free asset the set of allocation vectors for the risky assets is

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\ \mathbf{f} \geq \mathbf{0} \right\}. \tag{1}$$

For long Markowitz portfolios with a risk-free asset the set of allocation vectors for the risky assets is

$$\Lambda^+ = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} \leq 1\,,\ \mathbf{f} \geq \mathbf{0} \right\}. \tag{2}$$

It is clear that $\Lambda \subset \Lambda^+$.

Leveraged Portfolios. For Markowitz portfolios with no risk-free asset and with a leverage limit $\ell \in [0,\infty)$ the set of allocation vectors for the risky assets is

$$\Pi_\ell = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\ |\mathbf{f}| \leq 1 + 2\ell \right\}. \tag{3}$$

For Markowitz portfolios with a risk-free asset and with a leverage limit $\ell \in [0,\infty)$ the set of allocation vectors for the risky assets is

$$\Pi_\ell^+ = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, |1 - \mathbf{1}^\mathsf{T}\mathbf{f}| + |\mathbf{f}| \leq 1 + 2\ell \right\}. \tag{4}$$

It is clear that $\Pi_\ell \subset \Pi_\ell^+$ for every $\ell \in [0,\infty)$. It is also clear that if $\ell, \ell' \in [0,\infty)$ then $\ell \leq \ell'$ implies that

$$\Pi_\ell \subset \Pi_{\ell'} \qquad \text{and} \qquad \Pi_\ell^+ \subset \Pi_{\ell'}^+ \,.$$

Finally, we saw earlier that

$$\Lambda = \Pi_0 \qquad \text{and} \qquad \Lambda^+ = \Pi_0^+ \,.$$

Unrestricted Portfolios. For Markowitz portfolios with no risk-free asset and no leverage limit the set of allocation vectors for the risky assets is

$$\Pi_\infty = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1 \right\}. \tag{5}$$

For Markowitz portfolios with a risk-free asset and with no leverage limit the set of allocation vectors for the risky assets is

$$\Pi_\infty^+ = \mathbb{R}^N . \tag{6}$$

It is clear that $\Pi_\infty \subset \Pi_\infty^+$. As the notation suggests, if $\ell \in [0, \infty)$ then

$$\Pi_\ell \subset \Pi_\infty \qquad \text{and} \qquad \Pi_\ell^+ \subset \Pi_\infty^+ .$$

Moreover, Π_∞ is the union of all the Π_ℓ over $\ell > 0$ and Π_∞^+ is the union of all the Π_ℓ^+ over $\ell > 0$. These models are easy to analyze because they have no inequality constraints.

Price Ratios of Assets. Given a share price history $\{s_i(d)\}_{d=0}^{D}$ for N risky assets indexed by $i = 1, \cdots, N$, we define the price ratio history $\{\rho_i(d)\}_{d=0}^{D}$ by

$$\rho_i(d) = \frac{s_i(d)}{s_i(d-1)} \quad \text{for every } i = 1, \cdots, N \text{ and } d = 1, \cdots, D\,.$$

Because return rates $r_i(d)$ were defined by

$$r_i(d) = D_{\mathrm{y}} \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = D_{\mathrm{y}} \left(\frac{s_i(d)}{s_i(d-1)} - 1 \right),$$

we see that price ratios are related to the return rates by

$$\rho_i(d) = 1 + \tfrac{1}{D_{\mathrm{y}}} r_i(d) \quad \text{for every } i = 1, \cdots, N \text{ and } d = 1, \cdots, D\,.$$

Because share prices typically do not change much on any trading day, most price ratios will be close to 1. Because each share price is positive, every price ratio is positive.

Value Ratios of Markowitz Portfolios. The Markowitz portfolios with no risk-free asset are specified by allocation vectors $\mathbf{f}$ that satisfy $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$. In Lecture 2 we saw that if this portfolio has value history $\{\pi(d)\}_{d=1}^{D}$ then its value ratio on trading day d is

$$\frac{\pi(d)}{\pi(d-1)} = \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}\,,$$

where $\boldsymbol{\rho}(d)$ is the N-vector of price ratios on day d, which is

$$\boldsymbol{\rho}(d) = \begin{pmatrix} \rho_1(d) \\ \vdots \\ \rho_N(d) \end{pmatrix}.$$

The Markowitz portfolios with a risk-free asset are specified by allocation vectors $\mathbf{f}$. Its value ratio on trading day d is

$$\frac{\pi(d)}{\pi(d-1)} = \left(1 + \tfrac{1}{D_{\mathsf{y}}}\mu_{\mathsf{rf}}\right)\left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}\,.$$

Solvent Portfolios. For solvent Markowitz portfolios with no risk-free asset the set of allocation vectors for the risky assets is

$$\Omega = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\, 0 < \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \;\; \forall d \right\}. \tag{7}$$

For solvent Markowitz portfolios with a risk-free asset the set of allocation vectors for the risky assets is

$$\Omega^+ = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, 0 < \left(1 + \tfrac{1}{D_\mathsf{y}}\mu_{\mathsf{rf}}\right)\left(1 - \mathbf{1}^\mathsf{T}\mathbf{f}\right) + \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \;\; \forall d \right\}. \tag{8}$$

It is clear that $\Omega \subset \Omega^+$. Earlier we saw that

$$\Lambda \subset \Omega \qquad \text{and} \qquad \Lambda^+ \subset \Omega^+ \,.$$

The relationships between Π_ℓ and Ω and between Π_ℓ^+ and Ω^+ are less clear when $\ell > 0$. We will identify these relationships with the help of a more refined set of portfolio models that are also built upon value ratios.

Bounded Value-Ratio Portfolios. For Markowitz portfolios with no risk-free asset and with value ratios bounded within $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega_{[\underline{\rho},\overline{\rho}]} = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1\,,\ \underline{\rho} \leq \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} \leq \overline{\rho}\ \ \forall d \right\}. \tag{9}$$

For Markowitz portfolios with a risk-free asset and with value ratios bounded within $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega^{+}_{[\underline{\rho},\overline{\rho}]} = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \underline{\rho} \leq \left(1 + \tfrac{1}{D_{\mathsf{y}}}\mu_{\mathsf{rf}}\right)\left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} \leq \overline{\rho}\ \ \forall d \right\}. \tag{10}$$

It is clear that $\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega^{+}_{[\underline{\rho},\overline{\rho}]}$ for every $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$. It is also clear that if $[\underline{\rho}, \overline{\rho}]$ and $[\underline{\rho}', \overline{\rho}']$ are subsets of $(0, \infty)$ then $[\underline{\rho}, \overline{\rho}] \subset [\underline{\rho}', \overline{\rho}']$ implies that

$$\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega_{[\underline{\rho}',\overline{\rho}']} \qquad \text{and} \qquad \Omega^{+}_{[\underline{\rho},\overline{\rho}]} \subset \Omega^{+}_{[\underline{\rho}',\overline{\rho}']}\,.$$

Finally, it is clear that each of these portfolios are solvent. Specifically, we have $\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega$ and $\Omega^{+}_{[\underline{\rho},\overline{\rho}]} \subset \Omega^{+}$ for every $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$.

Bounded Below Value-Ratio Portfolios. For Markowitz portfolios with no risk-free asset and with value ratios bounded below by $\rho \in (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega_\rho = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\ \rho \leq \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}\ \ \forall d \right\}. \tag{11}$$

For Markowitz portfolios with a risk-free asset and with value ratios bounded below by $\rho \in (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega_\rho^+ = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \rho \leq \left(1 + \tfrac{1}{D_\mathsf{y}}\mu_\mathsf{rf}\right)\left(1 - \mathbf{1}^\mathsf{T}\mathbf{f}\right) + \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}\ \ \forall d \right\}. \tag{12}$$

It is clear that $\Omega_\rho \subset \Omega_\rho^+$ for every $\rho \in (0, \infty)$. It is also clear that if $\rho, \rho' \in (0, \infty)$ then $\rho \leq \rho'$ implies that

$$\Omega_\rho \subset \Omega_{\rho'} \qquad \text{and} \qquad \Omega_\rho^+ \subset \Omega_{\rho'}^+ \,.$$

Finally, it is clear that each of these portfolios are solvent. Specifically, we have $\Omega_\rho \subset \Omega$ and $\Omega_\rho^+ \subset \Omega^+$ for every $\rho \in (0, \infty)$.

Relationships with Leveraged Portfolios. Here we will give conditions that characterize when $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$, when $\Pi_\ell \subset \Omega_\rho$, and when $\Pi_\ell \subset \Omega$. We will use the long-short decomposition

$$\mathbf{f} = \mathbf{f}_+ - \mathbf{f}_- \,,$$

where $\mathbf{f}_+$ and $\mathbf{f}_-$ are uniquely determined by the conditions

$$\mathbf{f}_+ \geq \mathbf{0}\,, \qquad \mathbf{f}_- \geq \mathbf{0}\,, \qquad \mathbf{f}_+^{\mathsf{T}}\mathbf{f}_- = 0\,.$$

Because

$$\mathbf{1}^{\mathsf{T}}\mathbf{f} = \mathbf{1}^{\mathsf{T}}\mathbf{f}_+ - \mathbf{1}^{\mathsf{T}}\mathbf{f}_- \,, \qquad |\mathbf{f}| = \mathbf{1}^{\mathsf{T}}\mathbf{f}_+ + \mathbf{1}^{\mathsf{T}}\mathbf{f}_- \,,$$

we see that

$$\mathbf{1}^{\mathsf{T}}\mathbf{f}_+ = \frac{|\mathbf{f}| + \mathbf{1}^{\mathsf{T}}\mathbf{f}}{2}\,, \qquad \mathbf{1}^{\mathsf{T}}\mathbf{f}_- = \frac{|\mathbf{f}| - \mathbf{1}^{\mathsf{T}}\mathbf{f}}{2}\,.$$

The worst and best performing risky assets on trading day d have price ratios given by

$$\begin{aligned} \rho_{\mathsf{mn}}(d) &= \min\Big\{\rho_i(d) \,:\, i = 1, \cdots, N\Big\}\,, \\ \rho_{\mathsf{mx}}(d) &= \max\Big\{\rho_i(d) \,:\, i = 1, \cdots, N\Big\}\,. \end{aligned} \tag{13}$$

We expect that $0 < \rho_{\mathsf{mn}}(d) < \rho_{\mathsf{mx}}(d)$ on every trading day.

Remark. On most trading days a large, well-balanced portfolio will have an asset that decreases in value and another asset that increases in value. For such days we will have

$$0 < \rho_{\mathsf{mn}}(d) < 1 < \rho_{\mathsf{mx}}(d)\,.$$

For small portfolios it is not uncommon for $0 < \rho_{\mathsf{mn}}(d) < \rho_{\mathsf{mx}}(d) < 1$ on days when the whole market moves down, or for $1 < \rho_{\mathsf{mn}}(d) < \rho_{\mathsf{mx}}(d)$ on days when the whole market moves up.

We see from the definitions of $\rho_{\mathsf{mn}}(d)$ and $\rho_{\mathsf{mx}}(d)$ given in (13) that $\boldsymbol{\rho}(d)$ satisfies the entrywise inequalities

$$\rho_{\mathsf{mn}}(d)\,\mathbf{1} \le \boldsymbol{\rho}(d) \le \rho_{\mathsf{mx}}(d)\,\mathbf{1}\,.$$

These inequalities will be equalities for those entries corresponding to the worst and best performing assets respectively.

Because $\mathbf{f}_{\pm} \ge \mathbf{0}$, the above entrywise inequalities yield the bounds

$$\rho_{\mathsf{mn}}(d)\,\mathbf{1}^{\mathsf{T}}\mathbf{f}_{\pm} \le \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}_{\pm} \le \rho_{\mathsf{mx}}(d)\,\mathbf{1}^{\mathsf{T}}\mathbf{f}_{\pm}\,. \tag{14}$$

These inequalities will be equalities when the only nonneutral positions are held in the worst and best performing assets respectively.

We see from (14) that for every $\mathbf{f} \in \Pi_\ell$ a lower bound for $\boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}$ is

$$
\begin{aligned}
\boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} &= \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}_+ - \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f}_- \\
&\geq \rho_{\mathsf{mn}}(d)\, \mathbf{1}^\mathsf{T}\mathbf{f}_+ - \rho_{\mathsf{mx}}(d)\, \mathbf{1}^\mathsf{T}\mathbf{f}_- \\
&= \rho_{\mathsf{mn}}(d)\, \frac{|\mathbf{f}| + 1}{2} - \rho_{\mathsf{mx}}(d)\, \frac{|\mathbf{f}| - 1}{2} \\
&= \frac{\rho_{\mathsf{mx}}(d) + \rho_{\mathsf{mn}}(d)}{2} - \frac{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}{2}\, |\mathbf{f}| \\
&\geq \frac{\rho_{\mathsf{mx}}(d) + \rho_{\mathsf{mn}}(d)}{2} - \frac{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}{2}\, (1 + 2\ell) \\
&= \rho_{\mathsf{mn}}(d) - \Big(\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)\Big)\, \ell\,.
\end{aligned}
$$

This lower bound will be greater than or equal to $\underline{\rho}$ if and only if

$$
\ell \leq \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}\,.
$$

We see from (14) that for every $\mathbf{f} \in \Pi_\ell$ an upper bound for $\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}$ is

$$\begin{aligned}
\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} &= \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}_+ - \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}_- \\
&\le \rho_{\mathsf{mx}}(d)\mathbf{1}^{\mathsf{T}}\mathbf{f}_+ - \rho_{\mathsf{mn}}(d)\mathbf{1}^{\mathsf{T}}\mathbf{f}_- \\
&= \rho_{\mathsf{mx}}(d)\,\frac{|\mathbf{f}|+1}{2} - \rho_{\mathsf{mn}}(d)\,\frac{|\mathbf{f}|-1}{2} \\
&= \frac{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}{2}\,|\mathbf{f}| - \frac{\rho_{\mathsf{mx}}(d) + \rho_{\mathsf{mn}}(d)}{2} \\
&\le \frac{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}{2}\,(1+2\ell) - \frac{\rho_{\mathsf{mx}}(d) + \rho_{\mathsf{mn}}(d)}{2} \\
&= \rho_{\mathsf{mx}}(d) + \Big(\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)\Big)\,\ell\,.
\end{aligned}$$

This upper bound will be less than or equal to $\overline{\rho}$ if and only if

$$\ell \le \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)}\,.$$

Therefore $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$ for some $\ell > 0$ if

$$\ell \leq \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \,, \, \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} \,.$$

This condition will be satisfied by some $\ell > 0$ if and only if $\underline{\rho}$ and $\overline{\rho}$ satisfy

$$\underline{\rho} < \min_d \left\{ \rho_{\mathsf{mn}}(d) \right\} \,, \qquad \max_d \left\{ \rho_{\mathsf{mx}}(d) \right\} < \overline{\rho} \,.$$

Conversely, if ℓ does not satisfy this condition because for some d either

$$\ell > \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \qquad \text{or} \qquad \ell > \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \,,$$

then we can construct an $\mathbf{f} \in \Pi_\ell$ such that either $\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} < \underline{\rho}$ in the first case by being short in a best performing asset and long in a worst performing asset, or $\overline{\rho} < \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}$ in the second case by being long in a best performing asset and short in a worst performing asset.

Similarly, $\Pi_\ell \subset \Omega_\rho$ for some $\ell > 0$ if

$$\ell \leq \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d) - \rho}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} .$$

This condition will be satisfied by some $\ell > 0$ if and only if ρ satisfies

$$\rho < \min_d \left\{ \rho_{\mathsf{mn}}(d) \right\} .$$

Conversely, if ℓ does not satisfy this condition because for some d

$$\ell > \frac{\rho_{\mathsf{mn}}(d) - \rho}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} ,$$

then we can construct an $\mathbf{f} \in \Pi_\ell$ such that $\boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f} < \rho$ by being short in a best performing asset and long in a worst performing asset.

Because Ω is the union of all the Ω_ρ, it follows that $\Pi_\ell \subset \Omega$ for some $\ell > 0$ if and only if $\ell \in [0, \ell_{\mathsf{sol}})$ where

$$\ell_{\mathsf{sol}} = \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} .$$

In other words, every portfolio in Π_ℓ is solvent if and only if $\ell \in [0, \ell_{\mathsf{sol}})$.

Remark. The bound on leverage ℓ_{sol} as can be expressed as

$$\ell_{\mathsf{sol}} = \frac{1}{\max_d \left\{ \frac{\rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mn}}(d)} \right\} - 1} . \tag{15}$$

It depends upon the ratios $\rho_{\mathsf{mx}}(d)/\rho_{\mathsf{mn}}(d)$ over the history considered. These ratios can be close to 1 on days when the entire market moves up or down by a substantial amount. They can be largest on days when the market does not make a major move. It is wise to consider a long history when computing this bound for your own portfolio.

When $\ell \in [0, \ell_{\mathsf{sol}})$ we can identify an interval $[\underline{\rho}, \overline{\rho}]$ such that $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$. Define

$$\rho_{\mathsf{mn}} = \min_d \Big\{ \rho_{\mathsf{mn}}(d) \Big\} \,, \qquad \rho_{\mathsf{mx}} = \max_d \Big\{ \rho_{\mathsf{mx}}(d) \Big\} \,.$$

These are respectively the lowest and highest price ratios achieved by any asset over the entire history considered. Recall that $\underline{\rho}$ and $\overline{\rho}$ must satisfy $\underline{\rho} < \rho_{\mathsf{mn}}$ and $\rho_{\mathsf{mx}} \leq \overline{\rho}$.

Fact. If $\ell \in [0, \ell_{\mathsf{sol}})$ then $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$ where

$$\underline{\rho} = \left(1 - \frac{\ell}{\ell_{\mathsf{sol}}} \right) \rho_{\mathsf{mn}} \,, \qquad \overline{\rho} = \left(1 + \frac{\ell}{1 + \ell_{\mathsf{sol}}} \right) \rho_{\mathsf{mx}} \,. \tag{16}$$

Moreover, because $\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega_{\underline{\rho}}$, we have $\Pi_\ell \subset \Omega_{\underline{\rho}}$.

Proof. Let $\ell \in [0, \ell_{\mathsf{sol}})$. Let $\underline{\rho}$ and $\overline{\rho}$ be given by (16). Then

$$\min_d \left\{ \frac{\rho_{\mathsf{mn}}(d) - \underline{\rho}}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} \geq \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d) - \left(1 - \frac{\ell}{\ell_{\mathsf{sol}}}\right) \rho_{\mathsf{mn}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\}$$

$$= \frac{\ell}{\ell_{\mathsf{sol}}} \min_d \left\{ \frac{\rho_{\mathsf{mn}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} = \ell\,.$$

Similarly,

$$\min_d \left\{ \frac{\overline{\rho} - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\} \geq \min_d \left\{ \frac{\left(1 + \frac{\ell}{1+\ell_{\mathsf{sol}}}\right) \rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\}$$

$$= \frac{\ell}{1 + \ell_{\mathsf{sol}}} \min_d \left\{ \frac{\rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d) - \rho_{\mathsf{mn}}(d)} \right\}$$

$$= \frac{\ell}{1 + \ell_{\mathsf{sol}}} \left(1 + \ell_{\mathsf{sol}}\right) = \ell\,.$$

Because

$$\min_{d}\left\{\frac{\rho_{\mathsf{mn}}(d)-\underline{\rho}}{\rho_{\mathsf{mx}}(d)-\rho_{\mathsf{mn}}(d)}\,,\,\frac{\overline{\rho}-\rho_{\mathsf{mx}}(d)}{\rho_{\mathsf{mx}}(d)-\rho_{\mathsf{mn}}(d)}\right\}\geq\ell\,,$$

we conclude that $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$. □

Remark. Generally there is an interval $[\underline{\rho},\overline{\rho}]$ such that $\Pi_\ell \subset \Omega_{[\underline{\rho},\overline{\rho}]}$ that is smaller than the one given by (16). However, if $\rho_{\mathsf{mn}}(d)$ is close to ρ_{mn} and $\rho_{\mathsf{mx}}(d)$ is close to ρ_{mx} on days when $\rho_{\mathsf{mx}}(d)/\rho_{\mathsf{mn}}(d)$ is close to its maximum then the values of $\underline{\rho}$ and $\overline{\rho}$ given by (16) will be near optimal.

Remark. It is natural to ask why an investor who maintains a long portfolio should care about bounds on leverage limits. The answer is that bounds on leverage limits can fall well before a market bubble collapses. During a bubble some investors will succumb to the temptation of taking highly leveraged positions. The most highly leveraged investors will be stressed when bounds on leverage limits fall. They may have to shed some of their position to cover their margins. This creates market volatility, which in turn can drive bounds on leverage limits down further. This can go on for quite a while before the market turns down — if it turns down. Observant long investors can use this time move into a more conservative position. It is wise to use short histories when computing these bounds for this purpose.