Portfolios that Contain Risky Assets

Portfolio Models 2.

Markowitz Portfolios

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Portfolio Models 2. Markowitz Portfolios

A 1952 paper by Harry Markowitz had enormous influence on the theory and practice of portfolio management and financial engineering ever since. It presented his doctoral dissertation work at the Unversity of Chicago, for which he was awarded the Nobel Prize in Economics in 1990. It was the first work to quantify how diversifying a portfolio can reduce its risk without changing its potential reward. It did this because it was the first work to use the covariance between different assets in an essential way.

Portfolios. We will consider portfolios in which an investor can hold one of three positions with respect to any risky asset. The investor can:

- hold a *long position* by owning shares of the asset;
- hold a *short position* by selling borrowed shares of the asset;
- hold a *neutral position* by doing neither of the above.

In order to keep things simple, we will not consider derivative assets.

Remark. We hold a short position by borrowing shares of an asset from a lender (usually our broker) and selling them immediately. If the share price subsequently goes down then we can buy the same number of shares and give them to the lender, thereby paying off our loan and profiting by the price difference minus transaction costs. Of course, if the price goes up then our lender can force us either to increase our collateral or to pay off the loan by buying shares at this higher price, thereby taking a loss that might be larger than the original value of the shares.

The value of any portfolio that holds $n_i(d)$ shares of asset i at the end of trading day d is

$$\pi(d) = \sum_{i=1}^{N} n_i(d) s_i(d) \, . \tag{1}$$

If we hold a long position in asset i then $n_i(d) > 0$. If we hold a short position in asset i then $n_i(d) < 0$. If we hold a neutral position in asset i then $n_i(d) = 0$. We will assume that $\pi(d) > 0$ for every d.

Markowitz carried out his analysis on a class of idealized portfolios that are each characterized by a set of real numbers $\{f_i\}_{i=1}^N$ that satisfy

$$\sum_{i=1}^{N} f_i = 1\,. \tag{2}$$

The portfolio picks $n_i(d)$ at the beginning at each trading day d so that

$$\frac{n_i(d)s_i(d-1)}{\pi(d-1)} = f_i\,, \tag{3}$$

where $n_i(d)$ need not be an integer. We call these *Markowitz portfolios.* The portfolio holds a long position in asset i if $f_i > 0$ and holds a short position if $f_i < 0$. If every f_i is nonnegative then f_i is the fraction of the portfolio's value held in asset i at the beginning of each day. A Markowitz portfolio will be *self-financing* if we neglect trading costs because

$$\sum_{i=1}^{N} n_i(d)\, s_i(d-1) = \pi(d-1)\,. \tag{4}$$

Portfolio Returns. Definition (1) of $\pi(d)$, the self-financing property (4), relationship (3) between $n_i(d)$ and f_i, and the definition of $r_i(d)$ imply that the return rate $r(d)$ of a Markowitz portfolio for trading day d is

$$\begin{aligned} r(d) &= D_{\mathrm{y}}\, \frac{\pi(d) - \pi(d-1)}{\pi(d-1)} \\ &= D_{\mathrm{y}}\, \sum_{i=1}^{N} \frac{n_i(d)s_i(d) - n_i(d)s_i(d-1)}{\pi(d-1)} \\ &= \sum_{i=1}^{N} \frac{n_i(d)s_i(d-1)}{\pi(d-1)}\, D_{\mathrm{y}}\, \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \sum_{i=1}^{N} f_i\, r_i(d)\,. \end{aligned}$$

The return rate $r(d)$ for the Markowitz portfolio characterized by $\{f_i\}_{i=1}^{N}$ is therefore simply the linear combination with coefficients f_i of the $r_i(d)$. *This relationship makes the class of Markowitz portfolios easy to analyze. It is the reason we will use Markowitz portfolios to model real portfolios.*

This relationship can be expressed in the compact form

$$r(d) = \mathbf{f}^{\mathsf{T}} \mathbf{r}(d) \,, \tag{5}$$

where $\mathbf{f}$ and $\mathbf{r}(d)$ are the N-vectors defined by

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} , \qquad \mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix} .$$

We call $\mathbf{f}$ the *allocation vector* or simply the *allocation* because it gives the relative allocation of wealth within the portfolio.

The constraint (2) can be expressed in the compact form

$$\mathbf{1}^{\mathsf{T}} \mathbf{f} = 1 \,, \tag{6}$$

where $\mathbf{1}$ denotes the N-vector with each entry equal to 1.

Portfolio Statistics. Recall that if we assign weights $\{w(d)\}_{d=1}^{D}$ to the trading days of a daily return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ then the N-vector of return rate means $\mathbf{m}$ and the $N \times N$-matrix of return rate covariances $\mathbf{V}$ can be expressed in terms of $\mathbf{r}(d)$ as

$$\mathbf{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix} = \sum_{d=1}^{D} w(d)\,\mathbf{r}(d)\,,$$

$$\mathbf{V} = \begin{pmatrix} v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{pmatrix} = \sum_{d=1}^{D} w(d)\,\Big(\mathbf{r}(d) - \mathbf{m}\Big)\,\Big(\mathbf{r}(d) - \mathbf{m}\Big)^{\mathsf{T}}\,.$$

The choices of the daily return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ and weights $\{w(d)\}_{d=1}^{D}$ specify the *calibration* of our models. *Ideally $\mathbf{m}$ and $\mathbf{V}$ should not be overly sensitive to these choices.*

Because $r(d) = \mathbf{f}^{\mathsf{T}}\mathbf{r}(d)$, the portfolio return rate mean μ and variance v for the Markowitz portfolio with allocation $\mathbf{f}$ are then given by

$$
\begin{aligned}
\mu &= \sum_{d=1}^{D} w(d)\, r(d) = \sum_{d=1}^{D} w(d)\, \mathbf{f}^{\mathsf{T}}\mathbf{r}(d) = \mathbf{f}^{\mathsf{T}} \left(\sum_{d=1}^{D} w(d)\, \mathbf{r}(d) \right) \\
&= \mathbf{f}^{\mathsf{T}}\mathbf{m}\,, \\
v &= \sum_{d=1}^{D} w(d) \Big(r(d) - \mu \Big)^2 = \sum_{d=1}^{D} w(d) \Big(\mathbf{f}^{\mathsf{T}}\mathbf{r}(d) - \mathbf{f}^{\mathsf{T}}\mathbf{m} \Big)^2 \\
&= \sum_{d=1}^{D} w(d) \Big(\mathbf{f}^{\mathsf{T}}\mathbf{r}(d) - \mathbf{f}^{\mathsf{T}}\mathbf{m} \Big) \Big(\mathbf{r}(d)^{\mathsf{T}}\mathbf{f} - \mathbf{m}^{\mathsf{T}}\mathbf{f} \Big) \\
&= \mathbf{f}^{\mathsf{T}} \left(\sum_{d=1}^{D} w(d) \Big(\mathbf{r}(d) - \mathbf{m} \Big) \Big(\mathbf{r}(d) - \mathbf{m} \Big)^{\mathsf{T}} \right) \mathbf{f} = \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,.
\end{aligned}
$$

Hence, $\mu = \mathbf{f}^{\mathsf{T}}\mathbf{m}$ and $v = \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$. Because $\mathbf{V}$ is positive definite, $v > 0$.

Remark. These simple formulas for μ and v are the reason that returns are preferred over growth rates when compiling statistics of markets. The simplicity of these formulas arises because the return rate $r(d)$ for the Markowitz portfolio specified by the allocation $\mathbf{f}$ depends linearly upon the vector $\mathbf{r}(d)$ of return rates for the individual assets as $r(d) = \mathbf{f}^{\mathsf{T}}\mathbf{r}(d)$. In contrast, the growth rates $x(d)$ of a Markowitz portfolio are given by

$$\begin{aligned} x(d) = D_{\mathsf{y}} \log\left(\frac{\pi(d)}{\pi(d-1)}\right) &= D_{\mathsf{y}} \log\Big(1 + \tfrac{1}{D_{\mathsf{y}}} r(d)\Big) \\ &= D_{\mathsf{y}} \log\Big(1 + \tfrac{1}{D_{\mathsf{y}}} \mathbf{f}^{\mathsf{T}}\mathbf{r}(d)\Big) = D_{\mathsf{y}} \log\left(1 + \tfrac{1}{D_{\mathsf{y}}} \sum_{i=1}^{N} f_i \, r_i(d)\right) \\ &= D_{\mathsf{y}} \log\left(1 + \sum_{i=1}^{N} f_i \left(e^{\frac{x_i(d)}{D_{\mathsf{y}}}} - 1\right)\right) = D_{\mathsf{y}} \log\left(\sum_{i=1}^{N} f_i \, e^{\frac{x_i(d)}{D_{\mathsf{y}}}}\right) . \end{aligned}$$

Because the $x(d)$ are not linear functions of the $x_i(d)$, averages of $x(d)$ over d are not simply expressed in terms of averages of $x_i(d)$ over d.

Long Portfolios. *Because the value of any portfolio with short positions has the potential to go negative, many investors refuse to hold a short position in any risky asset.* Such investors will hold either a long or neutral position in each risky asset. Portfolios that hold no short positions are called *long portfolios*.

A Markowitz portfolio is long if and only if $f_i \geq 0$ for every i. This condition can be expressed compactly as

$$\mathbf{f} \geq \mathbf{0}\,, \tag{7}$$

where $\mathbf{0}$ denotes the N-vector with each entry equal to 0 and the inequality is understood entrywise. Therefore the set of all long Markowitz portfolios Λ is given by

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1\,,\; \mathbf{f} \geq \mathbf{0} \right\}. \tag{8}$$

Let $\mathbf{e}_i$ denote the vector whose i^{th} entry is 1 while every other entry is 0. For every $\mathbf{f} \in \Lambda$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where $f_i \geq 0$ for every $i = 1, \cdots, N$ and

$$\sum_{i=1}^{N} f_i = \mathbf{1}^{\mathsf{T}} \mathbf{f} = 1 \,.$$

This shows that Λ is simply all convex combinations of the vectors $\{\mathbf{e}_i\}_{i=1}^{N}$. We can visualize Λ when N is small.

When $N = 2$ it is the line segment that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,.$$

When $N = 3$ it is the triangle that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $N = 4$ it is the tetrahedron that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For general N it is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$.

Solvent Portfolios. Recall the return rate $r(d)$ of a portfolio for day d is

$$r(d) = D_{\mathrm{y}} \, \frac{\pi(d) - \pi(d-1)}{\pi(d-1)} \,,$$

where $\pi(d)$ is the value of the portfolio at the close of day d. Notice that $r(d) > 0$ when $\pi(d) < \pi(d-1) < 0$, which shows that it is a terrible notion of reward when $\pi(d-1) < 0$! Therefore we require that

$$\pi(d) > 0 \quad \text{for every } d = 0, \cdots, D \,.$$

In other words, we require that the portfolio to stay *solvent* over the given return rate history $\{\mathbf{r}(d)\}_{d=1}^{D}$.

We now use the fact that

$$\pi(d) = \pi(d-1)\left(1 + \tfrac{1}{D_{\mathrm{y}}} r(d)\right) \quad \text{for every } d = 1, \cdots, D \,.$$

This fact implies that

$$\pi(d) = \pi(0) \prod_{d'=1}^{d} \left(1 + \tfrac{1}{D_{\mathrm{y}}} r(d')\right) \quad \text{for every } d = 1, \cdots, D\,.$$

We see that the portfolio will be solvent over the given return rate history if and only if $\pi(0) > 0$ and

$$1 + \tfrac{1}{D_{\mathrm{y}}} r(d) > 0 \quad \text{for every } d = 1, \cdots, D\,.$$

For the Markowitz portfolio with allocation $\mathbf{f}$ we have $r(d) = \mathbf{f}^{\mathsf{T}} \mathbf{r}(d)$, so this solvency condition becomes

$$1 + \tfrac{1}{D_{\mathrm{y}}} \mathbf{f}^{\mathsf{T}} \mathbf{r}(d) > 0 \quad \text{for every } d = 1, \cdots, D\,. \tag{9}$$

Therefore the set of all solvent Markowitz portfolios Ω is given by

$$\Omega = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathsf{T}} \mathbf{f} = 1\,,\; 1 + \tfrac{1}{D_{\mathrm{y}}} \mathbf{f}^{\mathsf{T}} \mathbf{r}(d) > 0 \;\; \forall d \right\}. \tag{10}$$

Now we will show that every long Markowitz portfolio is solvent. In other words, we will show that $\Lambda \subset \Omega$.

Proof. Consider the N-vectors $\mathbf{1} + \frac{1}{D_\mathrm{y}}\mathbf{r}(d)$. Their i^{th} entry is

$$1 + \tfrac{1}{D_\mathrm{y}} r_i(d) = 1 + \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \frac{s_i(d)}{s_i(d-1)} > 0\,.$$

Therefore $\mathbf{1} + \frac{1}{D_\mathrm{y}}\mathbf{r}(d) > \mathbf{0}$ for every $d = 1, \cdots, D$.

Now let $\mathbf{f} \in \Lambda$. Because $\mathbf{f}^{\mathsf{T}}\mathbf{1} = 1$, $\mathbf{f} \geq \mathbf{0}$, and $\mathbf{1} + \mathbf{r}(d) > \mathbf{0}$, and we have

$$\begin{aligned} 1 + \tfrac{1}{D_\mathrm{y}}\mathbf{f}^{\mathsf{T}}\mathbf{r}(d) &= \mathbf{f}^{\mathsf{T}}\mathbf{1} + \tfrac{1}{D_\mathrm{y}}\mathbf{f}^{\mathsf{T}}\mathbf{r}(d) \\ &= \mathbf{f}^{\mathsf{T}}\Big(\mathbf{1} + \tfrac{1}{D_\mathrm{y}}\mathbf{r}(d)\Big) > 0 \quad \text{for every } d = 1, \cdots, D\,. \end{aligned}$$

We conclude that $\mathbf{f} \in \Omega$. Therefore $\Lambda \subset \Omega$.

Remark. The above proof gives a geometric way to think about the set Ω. We see from definition (10) that the set Ω is the intersection of the hyperplane $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$ with the half spaces

$$\left(\mathbf{1} + \frac{1}{D_{\mathrm{y}}}\mathbf{r}(d)\right)^{\mathsf{T}}\mathbf{f} > 0\,, \quad \text{for } d = 1, \cdots, D\,.$$

Because the entries of $\frac{1}{D_{\mathrm{y}}}\mathbf{r}(d)$ usually have absolute value much less than 1, the collection of vectors $\{\mathbf{1} + \frac{1}{D_{\mathrm{y}}}\mathbf{r}(d)\}_{d=1}^{D}$ can be thought of as a cloud of vectors clustered around the vector $\mathbf{1}$. The boundary of the half space $(\mathbf{1} + \frac{1}{D_{\mathrm{y}}}\mathbf{r}(d))^{\mathsf{T}}\mathbf{f} > 0$ is the hyperplane $(\mathbf{1} + \frac{1}{D_{\mathrm{y}}}\mathbf{r}(d))^{\mathsf{T}}\mathbf{f} = 0$, which is the hyperplane that is orthogonal to the vector $\mathbf{1} + \frac{1}{D_{\mathrm{y}}}\mathbf{r}(d)$.

Critique. Aspects of Markowitz portfolios are unrealistic. These include:

- the fact portfolios can contain fractional shares of any asset;
- the fact portfolios are rebalanced every trading day;
- the fact transaction costs and taxes are neglected;
- the fact dividends are neglected.

By making these simplifications the subsequent analysis becomes easier. The idea is to find the Markowitz portfolio that is best for a given investor. The expectation is that any real portfolio with an allocation close to that for the optimal Markowitz portfolio will perform nearly as well. Consequently, most investors rebalance at most a few times per year, and not every asset is involved each time. Transaction costs and taxes are thereby limited. Similarly, borrowing costs are kept to a minimum by not borrowing often. The model can be modified to account for dividends.

Remark. Portfolios of risky assets can be designed for *trading or investing.*

Traders often take positions that require constant attention. They might buy and sell assets on short time scales in an attempt to profit from market fluctuations. They might also take highly leveraged positions that expose them to enormous gains or loses depending how the market moves. They must be ready to handle margin calls. Trading is often a full time job.

Investors operate on longer time scales. They buy or sell an asset based on their assessment of its fundamental value over time. Investing does not have to be a full time job. Indeed, most people who hold risky assets are investors who are saving for retirement. Lured by the promise of high returns, sometimes investors will buy shares in funds designed for traders. At that point they have become gamblers, whether they realize it or not.

The ideas presented in these lectures are designed to balance investment portfolios, not trading portfolios.

Exercise. Compute μ and v based on daily data for the Markowitz portfolio with value equally distributed among the assets in each of the following groups:

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Exercise. The volatility of a portfolio is $\sigma = \sqrt{v}$. In the $\sigma\mu$-plane plot (σ, μ) for the two 2007 portfolios and (σ_i, m_i) for each of the 2007 assets in the previous exercise.

Exercise. In the $\sigma\mu$-plane plot (σ, μ) for the two 2009 portfolios and (σ_i, m_i) for each of the 2009 assets in the first exercise above.