Portfolios that Contain Risky Assets

Portfolio Models 9.

Long Portfolios with a Safe Investment

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We now consider investors who will not hold a short position in *any asset*. Such an investor will not borrow to invest in a risky asset, so *the safe investment is the only risky-free asset that we need to consider.* We will use the capital allocation line construction to obtain the *efficient long frontier* for long portfolios that might include the safe investment.

Efficient Long Frontier. We assume that the long frontier has already been constructed, and is given by $\sigma = \sigma_{\mathsf{lf}}(\mu)$ for $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$, where

$$\mu_{\mathsf{mn}} = \min\{m_i \,:\, i = 1, \cdots, N\}\,,$$
$$\mu_{\mathsf{mx}} = \max\{m_i \,:\, i = 1, \cdots, N\}\,.$$

We will assume that $\mu_{\mathsf{si}} < \mu_{\mathsf{mx}}$, because otherwise the safe investment is more efficient than any portfolio of risky assets.

The capital allocation line between the safe investment and the portfolio on the long frontier with return μ is the line segment in the $\sigma\mu$-plane between the points $(0, \mu_{\mathsf{si}})$ and $(\sigma_{\mathsf{lf}}(\mu), \mu)$. The slope of this line segment is

$$\nu_{\mathsf{ca}}(\mu) = \frac{\mu - \mu_{\mathsf{si}}}{\sigma_{\mathsf{lf}}(\mu)} \,.$$

The efficient long frontier is obtained by finding the capital allocation line with the greatest slope. In other words, we want to solve

$$\mu_{\mathsf{st}} = \arg\max\Big\{\nu_{\mathsf{ca}}(\mu) \;:\; \mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]\Big\} \,. \tag{1}$$

We set $\nu_{\mathsf{st}} = \nu_{\mathsf{ca}}(\mu_{\mathsf{st}})$ and $\sigma_{\mathsf{st}} = \sigma_{\mathsf{lf}}(\mu_{\mathsf{st}})$. The efficient long frontier is then given by $\mu = \mu_{\mathsf{ef}}(\sigma)$ where

$$\mu_{\mathsf{ef}}(\sigma) = \begin{cases} \mu_{\mathsf{si}} + \nu_{\mathsf{st}}\sigma & \text{for } \sigma \in [0, \sigma_{\mathsf{st}}] \,, \\ \sigma_{\mathsf{lf}}^{-1}(\sigma) & \text{for } \sigma \in [\sigma_{\mathsf{st}}, \sigma_{\mathsf{mx}}] \,, \end{cases} \tag{2}$$

where $\sigma \mapsto \sigma_{\mathsf{lf}}^{-1}(\sigma)$ is the inverse function of $\mu \mapsto \sigma_{\mathsf{lf}}(\mu)$.

Let us consider the maximization problem given in (1). Recall that the function $\mu \mapsto \sigma_{\mathsf{lf}}(\mu)$ is positive and continuous over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. This implies that the function $\mu \mapsto \nu_{\mathsf{ca}}(\mu)$ is continuous over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$, which implies that it has a maximum over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Because $\mu_{\mathsf{si}} < \mu_{\mathsf{mx}}$ we see that

$$\nu_{\mathsf{ca}}(\mu_{\mathsf{mx}}) = \frac{\mu_{\mathsf{mx}} - \mu_{\mathsf{si}}}{\sigma_{\mathsf{lf}}(\mu_{\mathsf{mx}})} = \frac{\mu_{\mathsf{mx}} - \mu_{\mathsf{si}}}{\sigma_{\mathsf{mx}}} > 0\,,$$

which implies that the maximum must be positive. Because the function $\mu \mapsto \sigma_{\mathsf{lf}}(\mu)$ is strictly convex over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$, the maximizer μ_{st} must be unique.

We will suppose that $\sigma'_{\mathsf{lf}}(\mu_{\mathsf{mx}}) > 0$ and that $\sigma'_{\mathsf{lf}}(\mu_{\mathsf{mn}}) \leq 0$, which is a common situation.

Efficient Long Frontier. The tangent line to the curve $\sigma = \sigma_{\mathsf{lf}}(\mu)$ at the point $(\sigma_{\mathsf{mx}}, \mu_{\mathsf{mx}})$ will intersect the μ-axis at $\mu = \eta_{\mathsf{mx}}$ where

$$\eta_{\mathsf{mx}} = \mu_{\mathsf{mx}} - \frac{\sigma_{\mathsf{lf}}(\mu_{\mathsf{mx}})}{\sigma'_{\mathsf{lf}}(\mu_{\mathsf{mx}})} \,.$$

We will consider the cases $\mu_{\mathsf{si}} \geq \eta_{\mathsf{mx}}$ and $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$ separately.

For the case when $\mu_{\text{si}} \geq \eta_{\text{mx}}$ we will make the additional assumption that $\mu_{\text{si}} < \mu_{\text{mx}}$. Then the efficient long frontier is simply given by

$$\mu_{\text{ef}}(\sigma) = \mu_{\text{si}} + \frac{\mu_{\text{mx}} - \mu_{\text{si}}}{\sigma_{\text{mx}}}\,\sigma \qquad \text{for } \sigma \in [0, \sigma_{\text{mx}}]\,.$$

Our additional assumption states that there is at least one risky asset that has a return mean greater than the return for the safe investment. This is usually the case. If it is not, the formula for $\mu_{\text{ef}}(\sigma)$ can be modified by appealing to the capital allocation line construction.

Remark. Notice that $\mu_{\text{ef}}(\sigma)$ given above is increasing over $\sigma \in [0, \sigma_{\text{mx}}]$. When $\mu_{\text{si}} = \mu_{\text{mx}}$ the capital allocation line construction would produce an expression for $\mu_{\text{ef}}(\sigma)$ that is constant, but might be defined over an interval larger than $[0, \sigma_{\text{mx}}]$. When $\mu_{\text{si}} > \mu_{\text{mx}}$ the capital allocation line construction would produce an expression for $\mu_{\text{ef}}(\sigma)$ that is decreasing over an interval larger than $[0, \sigma_{\text{mx}}]$.

For the case when $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$ there is a frontier portfolio $(\sigma_{\mathsf{st}}, \mu_{\mathsf{st}})$ such that the capital allocation between it and $(0, \mu_{\mathsf{si}})$ lies above the efficient long frontier. This means that $\mu_{\mathsf{st}} > \mu_{\mathsf{si}}$ and

$$\frac{\mu - \mu_{\mathsf{si}}}{\mu_{\mathsf{st}} - \mu_{\mathsf{si}}} \, \sigma_{\mathsf{st}} \leq \sigma_{\mathsf{lf}}(\mu) \quad \text{for every } \mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}] \, .$$

Because $\sigma_{\mathsf{lf}}(\mu)$ is an increasing, continuous function over $[\mu_{\mathsf{st}}, \mu_{\mathsf{mx}}]$ with image $[\sigma_{\mathsf{st}}, \sigma_{\mathsf{mx}}]$, it has an increasing, continuous inverse function $\sigma_{\mathsf{lf}}^{-1}(\sigma)$ over $[\sigma_{\mathsf{st}}, \sigma_{\mathsf{mx}}]$ with image $[\mu_{\mathsf{st}}, \mu_{\mathsf{mx}}]$. The efficient long frontier is then given by

$$\mu_{\mathsf{ef}}(\sigma) = \begin{cases} \mu_{\mathsf{si}} + \dfrac{\mu_{\mathsf{st}} - \mu_{\mathsf{si}}}{\sigma_{\mathsf{st}}} \, \sigma & \text{for } \sigma \in [0, \sigma_{\mathsf{st}}] \, , \\ \sigma_{\mathsf{lf}}^{-1}(\sigma) & \text{for } \sigma \in [\sigma_{\mathsf{st}}, \sigma_{\mathsf{mx}}] \, . \end{cases}$$

Because $\sigma_{\mathsf{lf}}(\mu)$ can be expressed as a list function, we can also express $\sigma_{\mathsf{lf}}^{-1}(\sigma)$ as a list function. We illustrate this below for the case $\mathbf{f}_{\mathsf{mv}} \geq \mathbf{0}$.

Suppose that $\mathbf{f}_{\mathsf{mv}} \geq \mathbf{0}$ and set $\overline{\mu}_0 = \mu_{\mathsf{mv}}$. Then $\sigma_{\mathsf{lf}}(\mu)$ has the form

$$\sigma_{\mathsf{lf}}(\mu) = \sigma_{\bar{\mathsf{f}}_k}(\mu) \equiv \sqrt{\sigma_{\mathsf{mv}_k}^2 + \left(\frac{\mu - \mu_{\mathsf{mv}_k}}{\nu_{\mathsf{as}_k}}\right)^2} \qquad \text{for } \mu \in [\overline{\mu}_k, \overline{\mu}_{k+1}]\,,$$

where σ_{mv_k}, μ_{mv_k}, and ν_{as_k} are the frontier parameters associated with the vector $\overline{\mathbf{m}}_k$ and matrix $\overline{\mathbf{V}}_k$ that determined $\sigma_{\bar{\mathsf{f}}_k}(\mu)$ in the k^{th} step of our iterative construction of $\sigma_{\mathsf{lf}}(\mu)$. In particular, $\sigma_{\mathsf{mv}_0} = \sigma_{\mathsf{mv}}$, $\mu_{\mathsf{mv}_0} = \mu_{\mathsf{mv}}$, and $\nu_{\mathsf{as}_0} = \nu_{\mathsf{as}}$ because $\overline{\mathbf{m}}_0 = \mathbf{m}$ and $\overline{\mathbf{V}}_0 = \mathbf{V}$.

Then $\sigma_{\mathsf{lf}}^{-1}(\sigma)$ has the form

$$\sigma_{\mathsf{lf}}^{-1}(\sigma) = \mu_{\mathsf{mv}_k} + \nu_{\mathsf{as}_k}\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_k}^2} \qquad \text{for } \sigma \in [\overline{\sigma}_k, \overline{\sigma}_{k+1}]\,,$$

where $\overline{\sigma}_k = \sigma_{\mathsf{lf}}(\overline{\mu}_k)$ and $\overline{\sigma}_{k+1} = \sigma_{\mathsf{lf}}(\overline{\mu}_{k+1})$.

Finally, we must find the tangency portfolio $(\sigma_{\mathsf{st}}, \mu_{\mathsf{st}})$. The tangent line to the long frontier at the point $(\overline{\sigma}_k, \overline{\mu}_k)$ intercepts the μ-axis at $\mu = \overline{\eta}_k$ where

$$\overline{\eta}_k = \overline{\mu}_k - \frac{\sigma_{\mathsf{lf}}(\overline{\mu}_k)}{\sigma'_{\mathsf{lf}}(\overline{\mu}_k)} = \mu_{\mathsf{mv}_k} - \frac{\nu_{\mathsf{as}_k}^2 \sigma_{\mathsf{mv}_k}^2}{\overline{\mu}_k - \mu_{\mathsf{mv}_k}}\,.$$

These intercepts satisfy $\overline{\eta}_k < \overline{\eta}_{k+1} \leq \eta_{\mathsf{mx}}$ when $\overline{\mu}_k < \overline{\mu}_{k+1} \leq \mu_{\mathsf{mx}}$. If we set $\overline{\eta}_0 = -\infty$ then for every $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$ there is a unique j such that

$$\overline{\eta}_j \leq \mu_{\mathsf{si}} < \overline{\eta}_{j+1}\,.$$

For this value of j we have the tangency parameters

$$\nu_{\mathsf{st}} = \nu_{\mathsf{as}_j}\sqrt{1 + \left(\frac{\mu_{\mathsf{mv}_j} - \mu_{\mathsf{si}}}{\nu_{\mathsf{as}_j}\,\sigma_{\mathsf{mv}_j}}\right)^2}\,, \quad \sigma_{\mathsf{st}} = \sigma_{\mathsf{mv}_j}\sqrt{1 + \left(\frac{\nu_{\mathsf{as}_j}\,\sigma_{\mathsf{mv}_j}}{\mu_{\mathsf{mv}_j} - \mu_{\mathsf{si}}}\right)^2}\,.$$

Therefore when $\mu_{\mathrm{si}} < \eta_{\mathrm{mx}}$ the efficient long frontier is given by

$$\mu_{\mathrm{ef}}(\sigma) = \begin{cases} \mu_{\mathrm{si}} + \nu_{\mathrm{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\mathrm{st}}]\,, \\ \mu_{\mathrm{mv}_j} + \nu_{\mathrm{as}_j}\sqrt{\sigma^2 - \sigma_{\mathrm{mv}_j}^2} & \text{for } \sigma \in [\sigma_{\mathrm{st}}, \overline{\sigma}_{j+1}]\,, \\ \mu_{\mathrm{mv}_k} + \nu_{\mathrm{as}_k}\sqrt{\sigma^2 - \sigma_{\mathrm{mv}_k}^2} & \text{for } \sigma \in [\overline{\sigma}_k, \overline{\sigma}_{k+1}] \text{ and } k > j\,. \end{cases}$$

Efficient Long Frontier Portfolios. Recall that the allocations associated with the efficient long frontier portfolios without the safe investment are given over $[\mu_{\mathrm{mv}}, \mu_{\mathrm{mx}}]$ by

$$\mathbf{f}_{\mathrm{lf}}(\mu) = \frac{\overline{\mu}_{k+1} - \mu}{\overline{\mu}_{k+1} - \overline{\mu}_k}\,\overline{\mathbf{f}}_k + \frac{\mu - \overline{\mu}_k}{\overline{\mu}_{k+1} - \overline{\mu}_k}\,\overline{\mathbf{f}}_{k+1} \quad \text{for } \mu \in [\overline{\mu}_k, \overline{\mu}_{k+1}]\,,$$

where $\overline{\mu}_0 = \mu_{\mathrm{mv}}$, $\overline{\mathbf{f}}_0 = \mathbf{f}_{\mathrm{mv}}$, while $\overline{\mathbf{f}}_k$ is the nodal portfolio allocation associated with $\overline{\mu}_k$ for any $k \geq 1$.

The return mean and allocation for the safe tangency portolio are

$$
\begin{aligned}
\mu_{\mathsf{st}} &= \mu_{\mathsf{mv}_j} + \frac{\nu_{\mathsf{as}_j}^2\,\sigma_{\mathsf{mv}_j}^2}{\mu_{\mathsf{mv}_j} - \mu_{\mathsf{si}}}\,,\\
\mathbf{f}_{\mathsf{st}} = \mathbf{f}_{\mathsf{lf}}(\mu_{\mathsf{st}}) &= \frac{\overline{\mu}_{j+1} - \mu_{\mathsf{st}}}{\overline{\mu}_{j+1} - \overline{\mu}_j}\,\overline{\mathbf{f}}_j + \frac{\mu_{\mathsf{st}} - \overline{\mu}_j}{\overline{\mu}_{j+1} - \overline{\mu}_j}\,\overline{\mathbf{f}}_{j+1}\,.
\end{aligned}
$$

The allocation of risky assets for any efficient long frontier portfolio is

$$
\mathbf{f}_{\mathsf{ef}}(\sigma) = \begin{cases}
\dfrac{\sigma}{\sigma_{\mathsf{st}}}\,\mathbf{f}_{\mathsf{st}} & \text{for } \sigma \in [0, \sigma_{\mathsf{st}}]\,,\\[2ex]
\dfrac{\overline{\mu}_{j+1} - \mu_{\mathsf{ef}}(\sigma)}{\overline{\mu}_{j+1} - \mu_{\mathsf{st}}}\,\mathbf{f}_{\mathsf{st}} + \dfrac{\mu_{\mathsf{ef}}(\sigma) - \mu_{\mathsf{st}}}{\overline{\mu}_{j+1} - \mu_{\mathsf{st}}}\,\overline{\mathbf{f}}_{j+1} & \text{for } \sigma \in [\sigma_{\mathsf{st}}, \overline{\sigma}_{j+1}]\,,\\[2ex]
\dfrac{\overline{\mu}_{k+1} - \mu_{\mathsf{ef}}(\sigma)}{\overline{\mu}_{k+1} - \overline{\mu}_k}\,\overline{\mathbf{f}}_k + \dfrac{\mu_{\mathsf{ef}}(\sigma) - \overline{\mu}_k}{\overline{\mu}_{k+1} - \overline{\mu}_k}\,\overline{\mathbf{f}}_{k+1} & \text{for } \sigma \in [\overline{\sigma}_k, \overline{\sigma}_{k+1}]\,,
\end{cases}
$$

where $k > j$ in the last case.

Remark. If we had also added a credit line to the portolio then we would have had to find the credit tangency portfolio and added the appropriate capital allocation line to the efficient long frontier. Typically there are two kinds of credit lines an investor might consider. One available from your broker usually requires that some of your risky assets be held as collateral. A downside of using this kind of credit line is that when the market goes down then your broker can force you either to add assets to your collateral or to sell assets in a low market to pay off your loan. Another kind of credit line might use real estate as collateral. Of course, if the price of real estate falls then you again might be forced to sell assets in a low market to pay off your loan. For investors who hold short positions in risky assets, these risks are hedged because they also make money when markets go down. *Investors who hold only long positions in risky assets and use a credit line can find themselves highly exposed to large losses in a market downturn.* It is not a wise position to take — yet many do in a bubble.

General Portfolio with Two Risky Assets. Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Here we will assume that $m_1 < m_2$, so that $\mu_{\mathsf{mn}} = m_1$ and $\mu_{\mathsf{mx}} = m_2$. The long frontier associated with just these two risky assets is given by

$$\sigma_{\mathsf{lf}}(\mu) = \sqrt{\sigma_{\mathsf{mv}}^2 + \left(\frac{\mu - \mu_{\mathsf{mv}}}{\nu_{\mathsf{as}}}\right)^2} \qquad \text{for } \mu \in [m_1, m_2],$$

where the frontier parameters are

$$\sigma_{\mathsf{mv}} = \sqrt{\frac{v_{11}v_{22} - v_{12}^2}{v_{11} + v_{22} - 2v_{12}}}, \qquad \nu_{\mathsf{as}} = \sqrt{\frac{(m_2 - m_1)^2}{v_{11} + v_{22} - 2v_{12}}},$$

$$\mu_{\mathsf{mv}} = \frac{(v_{22} - v_{12})m_1 + (v_{11} - v_{12})m_2}{v_{11} + v_{22} - 2v_{12}}.$$

The minimum volatility portfolio is

$$\mathbf{f}_{\mathsf{mv}} = \frac{1}{v_{11} + v_{22} - 2v_{12}} \begin{pmatrix} v_{22} - v_{12} \\ v_{11} - v_{12} \end{pmatrix} .$$

We will assume that $v_{12} \leq v_{11}$ and $v_{12} \leq v_{22}$, so that $\mathbf{f}_{\mathsf{mv}} \geq \mathbf{0}$ and $\mu_{\mathsf{mv}} = \mathbf{f}_{\mathsf{mv}}^{\mathsf{T}}\mathbf{m} \in [m_1, m_2]$. The efficient long frontier associated with just these two risky assets is then given by $(\sigma_{\mathsf{lf}}(\mu), \mu)$ where $\mu \in [\mu_{\mathsf{mv}}, m_2]$. We now show how this is modified by the inclusion of a safe investment.

The parameters associated with the construction of $\sigma_{\mathsf{lf}}(\mu)$ are

$$\begin{aligned} \bar{\mu}_0 &= \mu_{\mathsf{mv}} \,, \qquad & \bar{\mu}_1 &= \mu_{\mathsf{mx}} = m_2 \,, \\ \bar{\sigma}_0 &= \sigma_{\mathsf{mv}} \,, \qquad & \bar{\sigma}_1 &= \sigma_{\mathsf{mx}} = \sigma_2 = \sqrt{v_{22}} \,. \end{aligned}$$

The μ-intercept of the tangent line through $(\sigma_{\mathsf{mx}}, \mu_{\mathsf{mx}}) = (\sigma_2, m_2)$ is

$$\bar{\eta}_{\mathsf{mx}} = \bar{\mu}_{\mathsf{mx}} - \frac{\sigma_{\mathsf{lf}}(\bar{\mu}_{\mathsf{mx}})}{\sigma_{\mathsf{lf}}'(\bar{\mu}_{\mathsf{mx}})} = m_2 - \frac{\nu_{\mathsf{as}}^2\, \sigma_2^2}{m_2 - \mu_{\mathsf{mv}}} = \frac{v_{22}m_1 - v_{12}m_2}{v_{22} - v_{12}} \,.$$

We will present the two cases that arise in order of increasing complexity: $\eta_{\mathrm{mx}} \leq \mu_{\mathrm{si}}$ and $\mu_{\mathrm{si}} < \eta_{\mathrm{mx}}$.

When $\eta_{\mathrm{mx}} \leq \mu_{\mathrm{si}}$ the efficient long frontier is determined by

$$\mu_{\mathrm{ef}}(\sigma) = \mu_{\mathrm{si}} + \frac{m_2 - \mu_{\mathrm{si}}}{\sigma_2}\,\sigma \quad \text{for } \sigma \in [0, \sigma_2]\,.$$

When $\mu_{\mathrm{si}} < \eta_{\mathrm{mx}}$ the tangency portfolio parameters are given by

$$\nu_{\mathrm{st}} = \nu_{\mathrm{mv}} \sqrt{1 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}}{\nu_{\mathrm{as}}\,\sigma_{\mathrm{mv}}}\right)^2}\,, \qquad \sigma_{\mathrm{st}} = \sigma_{\mathrm{mv}} \sqrt{1 + \left(\frac{\nu_{\mathrm{as}}\,\sigma_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}}\right)^2}\,,$$

and the efficient long frontier is determined by

$$\mu_{\mathrm{ef}}(\sigma) = \begin{cases} \mu_{\mathrm{si}} + \nu_{\mathrm{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\mathrm{st}}]\,, \\ \mu_{\mathrm{mv}} + \nu_{\mathrm{as}} \sqrt{\sigma^2 - \sigma_{\mathrm{mv}}^2} & \text{for } \sigma \in [\sigma_{\mathrm{st}}, \sigma_2]\,. \end{cases}$$

Simple Portfolio with Three Risky Assets. Recall the portfolio of three risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m - d \\ m \\ m + d \end{pmatrix}, \qquad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

The efficient long frontier associated with just these three risky assets is given by $(\sigma_{\mathsf{lf}}(\mu), \mu)$ where $\mu \in [m, m + d]$ and

$$\sigma_{\mathsf{lf}}(\mu) = \begin{cases} s\sqrt{\dfrac{1+2r}{3} + \dfrac{1-r}{2}\left(\dfrac{\mu - m}{d}\right)^2} & \text{for } \mu \in [m, m + \frac{2}{3}d], \\ s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\dfrac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m + \frac{2}{3}d, m + d]. \end{cases}$$

We now show how this is modified by including a safe investment.

In the construction of $\sigma_{\mathsf{lf}}(\mu)$ we found that

$$\overline{\mu}_0 = m\,, \qquad \overline{\mu}_1 = m + \tfrac{2}{3}d\,, \qquad \overline{\mu}_2 = \mu_{\mathsf{mx}} = m + d\,,$$

$$\overline{\sigma}_0 = s\sqrt{\frac{1+2r}{3}}\,, \qquad \overline{\sigma}_1 = s\sqrt{\frac{5+4r}{9}}\,, \qquad \overline{\sigma}_2 = \sigma_{\mathsf{mx}} = s\,.$$

The frontier parameters for $\sigma_{\mathsf{f}_0}(\mu)$ were

$$\sigma_{\mathsf{mv}_0} = s\sqrt{\frac{1+2r}{3}}\,, \qquad \mu_{\mathsf{mv}_0} = m\,, \qquad \nu_{\mathsf{as}_0} = \frac{d}{s}\sqrt{\frac{2}{1-r}}\,,$$

while those for $\sigma_{\mathsf{f}_1}(\mu)$ were

$$\sigma_{\mathsf{mv}_1} = s\sqrt{\frac{1+r}{2}}\,, \qquad \mu_{\mathsf{mv}_1} = m + \tfrac{1}{2}d\,, \qquad \nu_{\mathsf{as}_1} = \frac{d}{2s}\sqrt{\frac{2}{1-r}}\,.$$

Because

$$\sigma_{\mathsf{lf}}(\mu)\sigma'_{\mathsf{lf}}(\mu) = s^2\frac{1-r}{2}\begin{cases}\dfrac{\mu-m}{d^2}\,, & \text{for } \mu \in \left[m, m+\tfrac{2}{3}d\right],\\[2ex] \dfrac{\mu-m-\frac{1}{2}d}{\frac{1}{4}d^2}\,, & \text{for } \mu \in \left[m+\tfrac{2}{3}d, m+d\right],\end{cases}$$

we can see that the μ-intercepts of the tangent lines through the points $(\overline{\sigma}_1, \overline{\mu}_1)$ and $(\overline{\sigma}_2, \overline{\mu}_2) = (\sigma_{\mathsf{mx}}, \mu_{\mathsf{mx}})$ are respectively

$$\overline{\eta}_1 = \overline{\mu}_1 - \frac{\sigma_{\mathsf{lf}}(\overline{\mu}_1)}{\sigma'_{\mathsf{lf}}(\overline{\mu}_1)} = m + \tfrac{2}{3}d - \frac{5+4r}{3-3r}\,d = m - \frac{1+2r}{1-r}\,d\,,$$

$$\eta_{\mathsf{mx}} = \mu_{\mathsf{mx}} - \frac{\sigma_{\mathsf{lf}}(\mu_{\mathsf{mx}})}{\sigma'_{\mathsf{lf}}(\mu_{\mathsf{mx}})} = m + d - \frac{1}{1-r}\,d = m - \frac{r}{1-r}\,d\,.$$

We will present the three cases that arise in order of increasing complexity: $\eta_{\mathsf{mx}} \leq \mu_{\mathsf{si}}$, $\overline{\eta}_1 \leq \mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$, and $\mu_{\mathsf{si}} < \overline{\eta}_1$.

When $\eta_{\mathsf{mx}} \leq \mu_{\mathsf{si}}$ the efficient long frontier is determined by

$$\mu_{\mathsf{ef}}(\sigma) = \mu_{\mathsf{si}} + \frac{\mu_{\mathsf{mx}} - \mu_{\mathsf{si}}}{\sigma_{\mathsf{mx}}}\,\sigma \quad \text{for } \sigma \in [0, \sigma_{\mathsf{mx}}]\,.$$

When $\overline{\eta}_1 \leq \mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$ the tangency portfolio parameters are given by

$$\nu_{\mathsf{st}} = \nu_{\mathsf{as}_1}\sqrt{1 + \left(\frac{\mu_{\mathsf{mv}_1} - \mu_{\mathsf{si}}}{\nu_{\mathsf{as}_1}\,\sigma_{\mathsf{mv}_1}}\right)^2}\,, \quad \sigma_{\mathsf{st}} = \sigma_{\mathsf{mv}_1}\sqrt{1 + \left(\frac{\nu_{\mathsf{as}_1}\,\sigma_{\mathsf{mv}_1}}{\mu_{\mathsf{mv}_1} - \mu_{\mathsf{si}}}\right)^2}\,,$$

and the efficient long frontier is determined by

$$\mu_{\mathsf{ef}}(\sigma) = \begin{cases} \mu_{\mathsf{si}} + \nu_{\mathsf{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\mathsf{st}}]\,, \\ \mu_{\mathsf{mv}_1} + \nu_{\mathsf{as}_1}\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_1}^2} & \text{for } \sigma \in [\sigma_{\mathsf{st}}, \sigma_{\mathsf{mx}}]\,. \end{cases}$$

When $\mu_{\mathsf{si}} < \overline{\eta}_1$ the tangency portfolio parameters are given by

$$\nu_{\mathsf{st}} = \nu_{\mathsf{as}_0}\sqrt{1 + \left(\frac{\mu_{\mathsf{mv}_0} - \mu_{\mathsf{si}}}{\nu_{\mathsf{as}_0}\,\sigma_{\mathsf{mv}_0}}\right)^2}\,, \quad \sigma_{\mathsf{st}} = \sigma_{\mathsf{mv}_0}\sqrt{1 + \left(\frac{\nu_{\mathsf{as}_0}\,\sigma_{\mathsf{mv}_0}}{\mu_{\mathsf{mv}_0} - \mu_{\mathsf{si}}}\right)^2}\,,$$

and the efficient long frontier is determined by

$$\mu_{\mathsf{ef}}(\sigma) = \begin{cases} \mu_{\mathsf{si}} + \nu_{\mathsf{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\mathsf{st}}]\,, \\ \mu_{\mathsf{mv}_0} + \nu_{\mathsf{as}_0}\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_0}^2} & \text{for } \sigma \in [\sigma_{\mathsf{st}}, \overline{\sigma}_1]\,, \\ \mu_{\mathsf{mv}_1} + \nu_{\mathsf{as}_1}\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_1}^2} & \text{for } \sigma \in [\overline{\sigma}_1, \sigma_{\mathsf{mx}}]\,. \end{cases}$$

Remark. The above formulas for $\mu_{\mathsf{ef}}(\sigma)$ can be made more explicit by replacing σ_{mv_0}, μ_{mv_0}, ν_{as_0}, σ_{mv_1}, μ_{mv_1}, ν_{as_1}, σ_{mx}, μ_{mx}, and $\overline{\sigma}_1$ with their explicit expressions in terms of m, d, s and r.