Portfolios that Contain Risky Assets

Portfolio Models 1.

Risk and Reward

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Math 420: *Mathematical Modeling*
March 21, 2016 version

Portfolios that Contain Risky Assets
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Portfolio Models 1. Risk and Reward

Suppose we are considering how to invest in N risky assets. Let $s_i(d)$ be the share price of the i^{th} asset at the close of the d^{th} trading day of a period that has D trading days. (Typically there are about 252 trading days in a year.) Here $s_i(0)$ is understood to be the share price at the close of the last trading day before the period being considered. We will assume that every $s_i(d)$ is positive. We would like to use the share price history $\{s_i(d)\}_{d=0}^{D}$ to gain insight into how to manage our portfolio over the coming period.

We will examine the following questions.

Can stochastic (random, probabilistic) models be built that quantitatively mimic this price history? Can such models be used to manage a portfolio?

Risky Assets. *The risk associated with an investment is the uncertainty of its outcome. Every investment has risk associated with it.* Hiding your cash under a mattress puts it at greater risk of loss to theft or fire than depositing it in a bank, and is a sure way to not make money. Depositing your cash into an FDIC insured bank account is the safest investment that you can make — the only risk of loss would be to an extreme national calamity. However, a bank account generally will yield a lower reward on your investment than any asset that has more risk associated with it. Such assets include stocks (equities), bonds, commodities (gold, oil, corn, etc.), private equity (venture capital), hedge funds, and real estate. With the exception of real estate, it is not uncommon for prices of these assets to fluctuate one to five percent in a day. Such assets are called *risky assets*.

Remark. *Market forces generally will insure that assets associated with higher potential reward are also associated with greater risk and vice versa.* Investment offers that seem to violate this principle are always scams.

We will consider three basic types of risky assets: *stocks*, *bonds*, and *commodities*. We will also consider *funds* that hold a combination of stocks, bonds, and/or commodities.

Stocks. Stocks (or equities) are part ownership of a company. Their value goes up when the company does well, and goes down when it does poorly. Some stocks pay a periodic (often quarterly) dividend of either cash or more stock. Stocks are traded on exchanges like the NYSE or NASDAQ.

The risk associated with a stock reflects the uncertainty about the future performance of the company. This uncertainty has many facets. For example, there might be questions about the future market share of its products, the availablity of the raw materials needed for its products, or the value of its current assets. Stocks in larger companies are generally less risky than stocks in smaller companies. *Stocks are generally higher reward/higher risk investments compared to bonds.*

Bonds. Bonds are loans to a government or company. The borrower usually makes a periodic (often quarterly) interest payment, and ultimately pays back the principle at a maturity date. Bonds are traded on secondary markets where their value is based on current interest rates. For example, if interest rates go up then bond values will go down on the secondary market.

The risk associated with a bond reflects the uncertainty about the credit worthiness of the borrower. Short term bonds are generally less risky than long term ones. Bonds from large entities are generally less risky than those from small entities. Bonds from governments are generally less risky than those from companies. (This is even true in some cases where the ratings given by some ratings agencies suggest otherwise.) *Bonds are generally lower reward/lower risk investments compared to stocks.*

Commodities. Commodities are hard assets such as gold, corn, oil, or real estate. They are bought in anticipation of their value going up. For example, this would be the case when an investor fears an inflationary period. Some commodities are bought in standard units (troy ounces, bushels, barrels). Others are bought through shares of a partnership. Some commodities like rental real estate will have regular income associated with it, but most provide no income.

The risk associated with a commodity reflects the uncertainty about the future demand for it and the future supply of it. This uncertainty has many facets because the variety of commodities is huge. For example, farm commodities are perishable, so will become worthless if they are held too long. The value of oil or gold will fall when new supplies are discovered. The demand for oil is generally higher during the northern hemisphere winter. Gold prices will spike during times of uncertainty, but tend to return to inflation adjusted levels. *Because of their variety, commodities can fall anywhere on the reward/risk spectrum.*

Mutual and Exchange-Traded Funds. These funds hold a combination of stocks, bonds, and/or commodities. Funds are set up by investment companies. Shares of mutual funds are bought and sold through the company that set it up. Shares of exchange-traded funds (ETFs) are bought and sold just as you would shares of a stock. *Funds of either type are typically lower reward/lower risk investments compared to the individual assets from which they are composed.*

Funds are managed in one of two ways: *actively* or *passively*. An actively-managed fund usually has a strategy to perform better than some market index like the S&P 500, Russell 1000, or Russell 2000. A passively-managed fund usually builds a portfolio so that its performance will match some market index, in which case it is called an *index fund*. Index funds are often portrayed to be *lower reward/lower risk* investments compared to actively-managed funds. *However, index funds typically will outperform most actively-managed funds.* Reasons for this include the facts that they have lower management fees and that they require smaller cash reserves.

Daily Returns. The first thing to understand that the share price of an asset has very little economic significance. This is because the size of our investment in an asset is the same if we own 100 shares worth 50 dollars each or 25 shares worth 200 dollars each. What is economically significant is how much our investment rises or falls in value. Because our investment in asset i would have changed by the price ratio $s_i(d)/s_i(d-1)$ over the course of day d, this ratio is economically significant. Rather than use the price ratio as the basic variable, it is traditional to use the so-called *daily return*, which is defined by

$$r_i(d) \equiv \frac{s_i(d)}{s_i(d-1)} - 1 = \frac{s_i(d) - s_i(d-1)}{s_i(d-1)}\,.$$

Therefore the share price of asset i increases on days when $r_i(d) > 0$ and decreases on days when $r_i(d) < 0$.

One way to understand daily returns is to set $r_i(d)$ equal to a constant μ. Upon solving the resulting relation for $s_i(d)$ we find that

$$s_i(d) = (1 + \mu)\, s_i(d - 1) \quad \text{for every } d = 1, \cdots, D\,.$$

By induction on d we can then derive the compound interest formula

$$s_i(d) = (1 + \mu)^d s_i(0) \quad \text{for every } d = 1, \cdots, D\,.$$

If we assume that $|\mu| << 1$ then we can use the approximation

$$\log(1 + \mu) \approx \mu\,,$$

whereby

$$s_i(d) \approx e^{\mu d}\, s_i(0)\,.$$

We thereby see μ is nearly the exponential growth rate in units of "per day" of the share price.

Share price histories can be gotten from websites like *Yahoo Finance* or *Google Finance*. For example, to compute the daily return history for Apple in 2010, type "Apple" into where is says "get quotes". You will see that Apple has the identifier AAPL and is listed on the NASDAQ. Click on "historical prices" and request share prices between "Dec 31, 2009" and "Dec 31, 2010". You will get a table that can be downloaded as a spreadsheet. The daily returns are computed using the *adjusted closing prices*.

We will consider N risky assets indexed by i. For each i we obtain the closing share price history $\{s_i(d)\}_{d=0}^{D}$. For each i we then compute the daily return history $\{r_i(d)\}_{d=1}^{D}$ by the formula

$$r_i(d) = \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} \,.$$

Because $r_i(d)$ depends upon $s_i(d-1)$, we will need the adjusted closing share price for day $d = 0$, which is the trading day just before day $d = 1$, the first trading day for which we want the daily return history.

Adjusted closing prices are used to compute daily returns in order to aviod complications that can arise if actual closing prices are used. For example, an asset might undergo a *stock split* or pay a *dividend*. Adjusted closing prices build the effect of such events into the historical price history.

Stock splits can happen after the share price of a stock has risen enough so that a single share might be too expensive to attract small investors. Let $n > m$. An n-to-m stock split will convert m shares of stock into n shares of stock. The share price will be reduced by a factor of $\frac{m}{n}$ so that the total capitalization remains unchanged. For example, if you own 50 shares of stock worth 75$ each then after a 3-to-1 stock split you will own 150 shares of stock worth 25$ each. Splits of 2-to-1, 3-to-1, and 3-to-2 are the most common, but other ratios are used too. Fractional shares are often converted to cash. For example, if you own 25 shares of stock worth 75$ each then after a 3-to-2 stock split you will own 37 shares of stock worth 50$ each plus get a 25$ cash payment. Reverse stock splits can also happen where $n < m$.

On the day of a n-to-m stock split the daily return should be computed from the actual closing price history as

$$r_i(d) = \frac{n s_i(d) - m s_i(d-1)}{m s_i(d-1)} \,.$$

The adjusted closing price history adjusts the share price history before the split so that we do not need to use this formula.

Dividends are used by companies to distribute profits to shareholders. They can be paid either in cash or in stock. They can be paid either at regular intervals or whenever a company chooses. How best to account for them depends on how they are being paid. The adjusted closing price history trys to adjust the share price history before the dividend in order to account for it. We will not describe how these adjustments are made.

Remark. Many assets do not split or pay dividends most years.

Remark. It is not obvious that daily returns are the right quantities upon which to build a theory of markets. For example, another possibility is to use the *growth rates* $x_i(d)$ defined by

$$x_i(d) = \log\left(\frac{s_i(d)}{s_i(d-1)}\right) .$$

These are also functions of the price ratio $s_i(d)/s_i(d-1)$. Moreover, they seem to be easier to understand than daily returns. For example, if we set $x_i(d)$ equal to a constant γ then by solving the resulting relation for $s_i(d)$ we find

$$s_i(d) = e^{\gamma}\, s_i(d-1) \quad \text{for every } d = 1, \cdots, D .$$

By induction on d we can show that

$$s_i(d) = e^{\gamma d} s_i(0) .$$

However, daily returns are favored because they have better properties with regard to porfolio statistics.

Insights from Simple Models. We can use models to gain understanding as well as to reflect reality. Here we use a simple (unrealistic) stochastic (random) model of returns to gain insights into aspects of markets.

Consider a market model in which each year every asset either goes up 20% or remains unchanged with equal probability. The return mean for this market is 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.2	1.2	1.44	1.2000
Second	1.2	1.0	1.20	1.0954
Third	1.0	1.2	1.20	1.0954
Fourth	1.0	1.0	1.00	1.0000

Three quarters of the investors see returns below 10%. One quarter of the investors have made no money.

The previous result did not look to bad for most investors, but perhaps you know that sometimes markets produce negative returns. Now consider a market model in which each year every asset either goes up 30% or goes down 10% with equal probability. The return mean for this market is also 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.3	1.3	1.69	1.3000
Second	1.3	0.9	1.17	1.0817
Third	0.9	1.3	1.17	1.0817
Fourth	0.9	0.9	0.81	0.9000

Three quarters of the investors see returns just above 8%. One quarter of the investors have lost money.

Next, consider a market model in which each year every asset either goes up 40% or goes down 20% with equal probability. The return mean for this market is also 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.4	1.4	1.96	1.4000
Second	1.4	0.8	1.12	1.0583
Third	0.8	1.3	1.12	1.0583
Fourth	0.8	0.8	0.64	0.8000

Three quarters of the investors see returns below 6%. One quarter of the investors have lost over one third of their investment.

Consider a market model in which each year every asset either goes up 50% or goes down 30% with equal probability. The return mean for this market is 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.5	1.5	2.25	1.5000
Second	1.5	0.7	1.05	1.0247
Third	0.7	1.5	1.05	1.0247
Fourth	0.7	0.7	0.49	0.7000

Three quarters of the investors see returns below 2.5%. One quarter of the investors have lost over half of their investment.

We see that fewer investors do well in more volatile markets. Therefore measures of volatility can be viewed a measures of risk.

There is another important insight to be gained from these models. Notice that if an individual investor buys an equal value of each asset in each of these markets then that investor sees a 10% return because

$$\frac{1.44 + 1.20 + 1.20 + 1.00}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,,$$
$$\frac{1.69 + 1.17 + 1.17 + 0.81}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,,$$
$$\frac{1.96 + 1.12 + 1.12 + 0.64}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,,$$
$$\frac{2.25 + 1.05 + 1.05 + 0.49}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,.$$

This shows the value of holding a diverse portfolio — specifically, that it reduces risk. We would like to understand how to apply this insight when designing real portfolios. This will require more realisitic models.

Statistical Approach. Daily returns $r_i(d)$ for asset i can vary wildly from day to day as the share price $s_i(d)$ rises and falls. Sometimes the reasons for such fluctuations are clear because they directly relate to some news about the company, agency, or government that issued the asset. For example, news of the Deepwater Horizon explosion on 20 April 2010 caused the share price of British Petroleum stock to fall. At other times they relate to news that benefit or hurt entire sectors of assets. For example, a rise in crude oil prices might benefit oil and railroad companies but hurt airline and trucking companies. And at yet other times they relate to general technological, demographic, or social trends. For example, new internet technology might benefit Google and Amazon (companies that exist because of the internet) but hurt traditional “brick and mortar” retailers. Finally, there is often no evident public reason for a particular stock price to rise or fall. The reason might be a takeover attempt, a rumor, insider information, or the fact a large investor needs cash for reasons unrelated to the asset.

Given the complexity of the dynamics underlying such market fluctuations, we adopt a statistical approach to quantifying their trends and correlations. More specifically, we will choose an appropriate set of statistics that will be computed from selected daily return histories of the relevant assets. We will then use these statistics to calibrate a model that will predict how a set of ideal portfolios might behave in the future.

The implicit assumption of this approach is that in the future the market will behave statistically as it did in the past. This means that the data should be drawn from a long enough daily return history to sample most of the kinds of market events that we expect to see in the future. However, the history should not be too long because very old data will not be relevant to the current market. *A common choice is to use the daily return history from the most recent twelve month period, which we dub "the past year".* For example, if we are planning our portfolio at the beginning of July 2015 then we might use the return histories for July 2014 through June 2015. Then D would be the number of trading days in this period.

Suppose that we have computed the daily return history $\{r_i(d)\}_{d=1}^{D}$ for each asset. At some point this data should be ported from the speadsheet into MATLAB, R, or another higher level environment that is well suited to the task ahead.

Mean-Variance Models. Next we compute the statistical quantities that we will use in our models: *means, variances, covariances,* and *correlations*.

The return *mean* for asset i over the past year, denoted m_i, is

$$m_i = \frac{1}{D} \sum_{d=1}^{D} r_i(d) \,.$$

This measures the trend of the share price. Unfortunately, it is commonly called the *expected return* for asset i even though it is *higher* than the return that most investors will see, especially in highly volatile markets. We will not use this misleading terminology.

The return *variance* for asset i, denoted v_i, is

$$v_i = \frac{1}{D} \sum_{d=1}^{D} \Big(r_i(d) - m_i \Big)^2 .$$

The return *standard deviation* for asset i over the year, denoted σ_i, is given by $\sigma_i = \sqrt{v_i}$. This is called the *volatility* of asset i. It measures the uncertainty of the market regarding the share price trend.

The *covariance* of the returns for assets i and j, denoted v_{ij}, is

$$v_{ij} = \frac{1}{D} \sum_{d=1}^{D} \Big(r_i(d) - m_i \Big)\Big(r_j(d) - m_j \Big) .$$

Notice that $v_{ii} = v_i$. The $N \times N$ matrix (v_{ij}) is symmetric and nonnegative definite. It will usually be positive definite — so we will assume it to be so.

Finally, the *correlation* of the returns for assets i and j, denoted c_{ij}, is

$$c_{ij} = \frac{v_{ij}}{\sigma_i \, \sigma_j} \, .$$

Notice that $-1 \leq c_{ij} \leq 1$. We say assets i and j are *positively correlated* when $0 < c_{ij} \leq 1$ and *negatively correlated* when $-1 \leq c_{ij} < 0$. Positively correlated assets will tend to move in the same direction, while negatively correlated ones will often move in opposite directions.

We will consider the N-vector of means (m_i) and the symmetric $N \times N$ matrix of covariances (v_{ij}) to be our basic statistical quantities. We will build our models to be consistent with these statistics. The variances (v_i), volatilities (σ_i), and correlations (c_{ij}) can then be easily obtained from (m_i) and (v_{ij}) by formulas that are given above. The computation of the statistics (m_i) and (v_{ij}) from the return histories is called the *calibration* of our models.

Remark. Here the trading day is an arbitrary measure of time. From a theoretical viewpoint we could equally well have used a shorter measure like half-days, hours, quarter hours, or minutes. The shorter the measure, the more data has to be collected and analyzed. This extra work is not worth doing unless you profit sufficiently. Alternatively, we could have used a longer measure like weeks, months, or quarters. The longer the measure, the less data you use, which means you have less understanding of the market. For many investors daily or weekly data is a good balance. If you use weekly data $\{s_i(w)\}_{w=0}^{52}$, where $s_i(w)$ is the share price of asset i at the end of week w, then the return of asset i for week w is

$$r_i(w) = \frac{s_i(w) - s_i(w-1)}{s_i(w-1)} \,.$$

You have to make consistent changes when computing m_i, v_i, and v_{ij} by replacing d with w in their defining formulas.

General Calibration. We can consider a history $\{r(d)\}_{d=1}^{D}$ over a period of D trading days and assign day d a weight $w(d) > 0$ such that the weights $\{w(d)\}_{d=1}^{D}$ satisfy

$$\sum_{d=1}^{D} w(d) = 1 \,.$$

The return means and covariances are then given by

$$m_i = \sum_{d=1}^{D} w(d)\, r_i(d) \,,$$
$$v_{ij} = \sum_{d=1}^{D} w(d) \Big(r_i(d) - m_i \Big)\Big(r_j(d) - m_j \Big) \,.$$

In practice the history can extend over a period of one to five years. There are many ways to choose the weights $\{w(d)\}_{d=1}^{D}$. The most common choice is the so-called *uniform weighting*; this gives each day the same weight by setting $w(d) = 1/D$. On the other hand, we might want to give more weight to more recent data. For example, we can give each trading day a positive weight that depends only on the quarter in which it lies, giving greater weight to more recent quarters. We could also consider giving different weights to different days of the week, but such a complication should be avoided unless it yields a clear benefit.

We will have greater confidence in m_i and v_{ij} when they are relatively insensitive to different choices of D and the weights $w(d)$. We can get an idea of the magnitude of this sensitivity by checking the robustness of m_i and v_{ij} to a range of such choices.