Optimizing over Interventions within a Budget

Brian Hunt
University of Maryland
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Budget Constraint

- Recall our 2-group SI model with type $a$ and $b$ interventions:

  \[
  \frac{dS_1}{dt} = -p_{11}S_1I_1 - p_{12}S_1I_2 - a_1S_1 \\
  \frac{dI_1}{dt} = p_{11}S_1I_1 + p_{12}S_1I_2 - (a_1 + b_1)I_1 \\
  \frac{dS_2}{dt} = -p_{21}S_2I_1 - p_{22}S_2I_2 - a_2S_2 \\
  \frac{dI_2}{dt} = p_{21}S_2I_1 + p_{22}S_1I_2 - (a_2 + b_2)I_2.
  \]

- The impact $M(a_1, a_2, b_1, b_2)$ (fraction of the initial at-risk population saved from infection by the intervention) is an increasing function of each parameter $a_1, a_2, b_1, b_2$.

- We now consider the problem of maximizing $M$ subject to a “budget” constraint $K(a_1, a_2, b_1, b_2) \leq K_{\text{max}}$ where $K(a_1, a_2, b_1, b_2)$ is the cost of achieving parameters $a_1, a_2, b_1, b_2$. 
Linear Budget Functions

- The simplest class of cost functions are linear functions
  \[ K(a_1, a_2, b_1, b_2) = c_{a1}a_1 + c_{a2}a_2 + c_{b1}b_1 + c_{b2}b_2 \]
  where the \( c \)'s are positive constants representing “marginal” costs.

- To simplify further, let’s assume \( c_{a1} = c_{a2} \) and \( c_{b1} = c_{b2} \) and let \( c = c_{a1}/c_{b1} = c_{a2}/c_{b2} \).

- Let’s normalize (choose units of cost) so that \( c_{b1} = c_{b2} = 1 \), and call the resulting cost function \( K_c \).

- The main flaw in a linear cost function is that it doesn’t have the “diminishing returns” observed in real life.
Constrained Optimization

• We are considering a constrained optimization problem; in addition to the constraint $K_c(a_1, a_2, b_1, b_2) \leq K_{\text{max}}$ we have $a_1, a_2, b_1, b_2 \geq 0$. These inequalities describe a 4-dimensional simplex over which we want to maximize the impact $M$.

• Since $M$ is an increasing function of the parameters $a_1, a_2, b_1,$ and $b_2$, the maximum impact will occur when the entire budget is used:
  
  $K_c(a_1, a_2, b_1, b_2) = K_{\text{max}}$. This equality allows one parameter to be determined from the other three, reducing the domain to a 3-dimensional simplex – a tetrahedron.

• The maximum of $M$ often occurs when one or more of the parameters is zero, meaning that the maximum occurs on one of the faces, edges, or vertices of the tetrahedron.
Optimization Strategy

- Iterative, “guess-and-perturb” optimization algorithms are problematic for constrained optimization because the allowed perturbations depend on whether one is inside the constraint domain or on its boundary, and where on the boundary.
- A simpler approach, feasible with a few parameters, is to search the entire domain to a certain resolution – choose a closely-spaced grid and search over the grid points in the domain.
- To distinguish between the boundary and the interior of the domain, you need to sample all parts of the boundary. For the tetrahedron described on the previous slide, a rectangular grid will sample the faces and edges that are aligned with the coordinate axes, but you may need to choose additional points to sample the diagonal face and edges.