Modeling Epidemics: Introduction, Simple Model, and Linear Least Squares

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First Models

- Preliminary goal: Model the spread of an contagious illness through a population.
- Simplifying assumptions:
  - The total population N is constant in time.
  - A newly infected person becomes infectious the next day and remains infectious forever.
  - Each infectious person is equally likely (probability p) to infect each noninfectious person on a given day.
- Let $\mathcal{I}(t)$ be the number of infectious people at the start of day t.

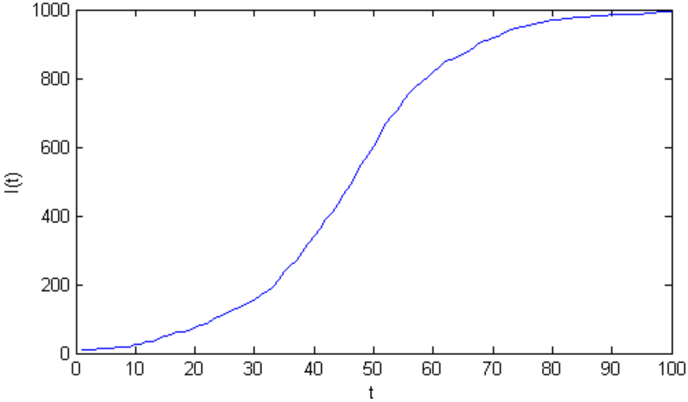
Stochastic Model

- Number the people from 1 to N.
- Let $x_n(t)$ be the infectious status (1 if infectious, 0 if not) of person n at the start of day t.
- We can simulate a possible spread of the illness with the following program (“`rand`”= random number):

```
for t=1:T-1
    for n=1:N
        let x(n,t+1)=x(n,t)
        for m=1:N
            if x(m,t)=1 and rand<p, then let x(n,t+1)=1
        end
    end
end
```

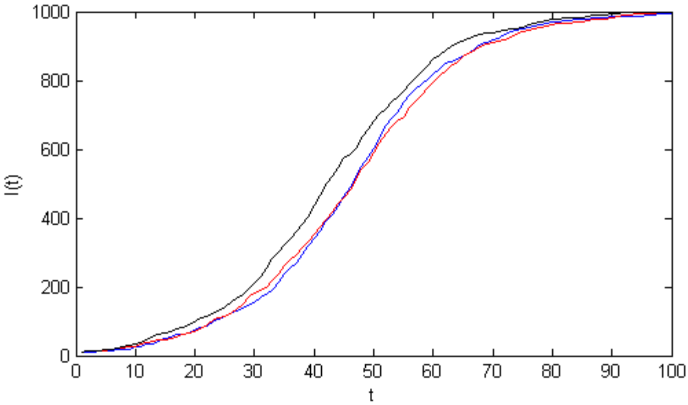
Simulation Results

- Notice that $\mathcal{I}(t) = \sum_{n=1}^{N} x_n(t)$.
- Here are the results of a simulation with $p = 10^{-4}$, $N = 1000$, and $\mathcal{I}(1) = 10$:



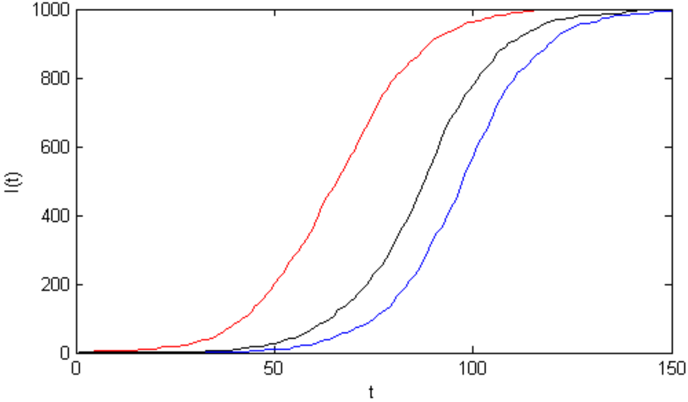
Simulation Results

- And here are the results of three different simulations with $p = 10^{-4}$, $N = 1000$, and $\mathcal{I}(1) = 10$:



Simulation Results

- Finally, here are the results of three different simulations with $p = 10^{-4}$, $N = 1000$, and $\mathcal{I}(1) = 1$:



Expected (Average) Daily Outcome

- Let's determine the expected number of people infected on a day t that starts with $\mathcal{I}(t)$ infectious people and $N - \mathcal{I}(t)$ who are susceptible to infection.
- A susceptible person n has probability $1 - p$ of NOT being infected on day t by a given infectious person m. Therefore, person n has probability $(1 - p)^{\mathcal{I}(t)}$ of NOT being infected on day t.
- The expected number of people who are infected on day t is then $[1 - (1 - p)^{\mathcal{I}(t)}][N - \mathcal{I}(t)]$, so

$$E[\mathcal{I}(t+1)] = \mathcal{I}(t) + [1 - (1 - p)^{\mathcal{I}(t)}][N - \mathcal{I}(t)]$$

Deterministic Models

- If both $\mathcal{I}(t)$ and $N - \mathcal{I}(t)$ are large enough, it may be reasonable to approximate $\mathcal{I}(t+1)$ by its expected value, resulting in a deterministic model:

$$\mathcal{I}(t+1) = \mathcal{I}(t) + [1 - (1-p)^{\mathcal{I}(t)}][N - \mathcal{I}(t)] \qquad (1)$$

- If $p\mathcal{I}(t)$ is small, we can approximate $(1-p)^{\mathcal{I}(t)}$ by $1 - p\mathcal{I}(t)$, yielding a simpler model:

$$\mathcal{I}(t+1) = \mathcal{I}(t) + p\mathcal{I}(t)[N - \mathcal{I}(t)] \qquad (2)$$

- For these models, given $\mathcal{I}(1)$ we can compute $\mathcal{I}(2)$, $\mathcal{I}(3)$, ...

Deterministic versus Stochastic

- These deterministic models are much more efficient to compute (1 calculation versus N^2 for the stochastic model). Their predictions may be just as reasonable as any particular simulation of the stochastic model.
- The stochastic model can give some idea of the uncertainty of its predictions via multiple simulations; the deterministic models we've written down say nothing about their uncertainty.

Continuous-Time Model

- The models discussed so far are called discrete-time models; time t takes on only integer values.
- When the quantities being modeled change slowly enough, we can approximate these models by continuous-time processes. Approximating model (2) by replacing $\Delta\mathcal{I} = \mathcal{I}(t+1) - \mathcal{I}(t)$ by $d\mathcal{I}/dt$, we get

$$d\mathcal{I}/dt = p\mathcal{I}(t)[N - \mathcal{I}(t)]. \tag{3}$$

  This differential equation is commonly called the Logistic Growth Model.
- We can write down an exact solution to this differential equation:

$$\mathcal{I}(t) = \frac{N\mathcal{I}(0)}{\mathcal{I}(0) + [N - \mathcal{I}(0)]e^{-pNt}}$$

Fitting the Model to Data

- The solution $\mathcal{I}(t)$ of model (3) has three parameters: N, p, and $\mathcal{I}(0)$. Suppose we know N but not the other two parameters. Given a set of data points $[t_j, \mathcal{I}_j]$, we can ask which values of p and $\mathcal{I}(0)$ best fit the data.
- [A more fundamental (but more difficult) question is whether the model can adequately fit the data at all; are there ANY parameters of the model that fit the data reasonably well?]
- We could try to minimize the sum of the squares of the residuals $\mathcal{I}_j - \mathcal{I}(t_j)$. However, this would be a nonlinear least squares problem, because $\mathcal{I}(t)$ does not depend linearly on p or $\mathcal{I}(0)$.

Method 1 to use Linear Least Squares

- If the data is given at consecutive values of t, say $t_j = j$, then one approach is to use model (2) and write
$$\mathcal{I}(t+1) - \mathcal{I}(t) = p\mathcal{I}(t)[N - \mathcal{I}(t)].$$
  The right-hand side is a linear function of the parameter p, and linear least squares yields the value of p that minimizes the sum of the squares of the residuals $\mathcal{I}_{j+1} - \mathcal{I}_j - p\mathcal{I}_j(N - \mathcal{I}_j)$.
- This doesn't resolve the question of which value of $\mathcal{I}(0)$ to use. If we let $t_0 = 0$ for the first data point, then we could let $\mathcal{I}(0) = \mathcal{I}_0$. However, this might not be the best choice of $\mathcal{I}(0)$ in order to make the residuals $\mathcal{I}_j - \mathcal{I}(t_j)$ small.

Method 2 to use Linear Least Squares

- Going back to the solution of model (3), we can make a transformation of variables so that the transformed solution does depend linearly on its parameters. First we divide both sides into N and simplify:

$$N/\mathcal{I}(t) = 1 + [N/\mathcal{I}(0) - 1]e^{-pNt}$$

- Next subtract 1 and take the logarithm:

$$\log[N/\mathcal{I}(t) - 1] = \log[N/\mathcal{I}(0) - 1] - pNt$$

- Let $Z(t) = \log[N/\mathcal{I}(t) - 1]$; then the model becomes $Z(t) = Z(0) - pNt$. This is a linear function of the parameters pN and $Z(0)$. One can transform the data to pairs (t_j, Z_j), use linear least squares to determine values for pN and $Z(0)$, and then solve for p and $\mathcal{I}(0)$.

Caveat

- Both methods of using linear least squares transform the model or its solution into a linear relationship between two quantities that can be computed from the data points $(t_j, \mathcal{I}_j)$; in the second method, the model predicts that Z_j is a linear function of t_j.
- Rather than simply accept the result of the least squares fit, one should graph the predicted relationship (e.g., Z_j versus t_j) and see if it actually looks linear. This gives some idea of how valid the model is.
- Regardless of how one determines values for p and $\mathcal{I}(0)$, one should also check directly how well the resulting $\mathcal{I}(t)$ fits the data.