Modeling Epidemics: Introduction, Simple Model, and Linear Least Squares

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First Models

- Preliminary goal: Model the spread of a contagious illness through a population.
- Simplifying assumptions:
  - The total population $N$ is constant in time.
  - A newly infected person becomes infectious the next day and remains infectious forever.
  - Each infectious person is equally likely (probability $p$) to infect each noninfectious person on a given day.
- Let $I(t)$ be the number of infectious people at the start of day $t$. 
Stochastic Model

- Number the people from 1 to \( N \).
- Let \( x_n(t) \) be the infectious status (1 if infectious, 0 if not) of person \( n \) at the start of day \( t \).
- We can simulate a possible spread of the illness with the following program ("rand" = random number):

```plaintext
for t=1:T-1
    for n=1:N
        let x(n,t+1)=x(n,t)
        for m=1:N
            if x(m,t)=1 and rand<p, then let x(n,t+1)=1
        end
    end
end
```
Simulation Results

- Notice that $I(t) = \sum_{n=1}^{N} x_n(t)$.
- Here are the results of a simulation with $\rho = 10^{-4}$, $N = 1000$, and $I(1) = 10$:
Simulation Results

- And here are the results of three different simulations with $p = 10^{-4}$, $N = 1000$, and $I(1) = 10$: 

![Simulation Results Graph](image)
Simulation Results

• Finally, here are the results of three different simulations with $\rho = 10^{-4}$, $N = 1000$, and $\mathcal{I}(1) = 1$:
Expected (Average) Daily Outcome

- Let’s determine the expected number of people infected on a day \( t \) that starts with \( I(t) \) infectious people and \( N - I(t) \) who are susceptible to infection.
- A susceptible person \( n \) has probability \( 1 - p \) of NOT being infected on day \( t \) by a given infectious person \( m \). Therefore, person \( n \) has probability \( (1 - p)^{I(t)} \) of NOT being infected on day \( t \).
- The expected number of people who are infected on day \( t \) is then \([1 - (1 - p)^{I(t)}][N - I(t)]\), so
  
  $$E[I(t + 1)] = I(t) + [1 - (1 - p)^{I(t)}][N - I(t)]$$
Deterministic Models

- If both $\mathcal{I}(t)$ and $N - \mathcal{I}(t)$ are large enough, it may be reasonable to approximate $\mathcal{I}(t + 1)$ by its expected value, resulting in a deterministic model:

  $$\mathcal{I}(t + 1) = \mathcal{I}(t) + [1 - (1 - p)^{\mathcal{I}(t)}][N - \mathcal{I}(t)] \quad (1)$$

- If $p\mathcal{I}(t)$ is small, we can approximate $(1 - p)^{\mathcal{I}(t)}$ by $1 - p\mathcal{I}(t)$, yielding a simpler model:

  $$\mathcal{I}(t + 1) = \mathcal{I}(t) + p\mathcal{I}(t)[N - \mathcal{I}(t)] \quad (2)$$

- For these models, given $\mathcal{I}(1)$ we can compute $\mathcal{I}(2)$, $\mathcal{I}(3)$, ....
• These deterministic models are much more efficient to compute (1 calculation versus $N^2$ for the stochastic model). Their predictions may be just as reasonable as any particular simulation of the stochastic model.

• The stochastic model can give some idea of the uncertainty of its predictions via multiple simulations; the deterministic models we’ve written down say nothing about their uncertainty.
Continuous-Time Model

- The models discussed so far are called discrete-time models; time $t$ takes on only integer values.
- When the quantities being modeled change slowly enough, we can approximate these models by continuous-time processes. Approximating model (2) by replacing $\Delta \mathcal{I} = \mathcal{I}(t + 1) - \mathcal{I}(t)$ by $d\mathcal{I}/dt$, we get
  \[ \frac{d\mathcal{I}}{dt} = p\mathcal{I}(t)[N - \mathcal{I}(t)]. \] (3)

  This differential equation is commonly called the Logistic Growth Model.
- We can write down an exact solution to this differential equation:
  \[ \mathcal{I}(t) = \frac{N\mathcal{I}(0)}{\mathcal{I}(0) + [N - \mathcal{I}(0)]e^{-pNt}}. \]
Fitting the Model to Data

- The solution $\mathcal{I}(t)$ of model (3) has three parameters: $N$, $p$, and $\mathcal{I}(0)$. Suppose we know $N$ but not the other two parameters. Given a set of data points $[t_j, \mathcal{I}_j]$, we can ask which values of $p$ and $\mathcal{I}(0)$ best fit the data.

- [A more fundamental (but more difficult) question is whether the model can adequately fit the data at all; are there ANY parameters of the model that fit the data reasonably well?]

- We could try to minimize the sum of the squares of the residuals $\mathcal{I}_j - \mathcal{I}(t_j)$. However, this would be a nonlinear least squares problem, because $\mathcal{I}(t)$ does not depend linearly on $p$ or $\mathcal{I}(0)$. 
Method 1 to use Linear Least Squares

- If the data is given at consecutive values of $t$, say $t_j = j$, then one approach is to use model (2) and write

$$I(t + 1) - I(t) = pI(t)[N - I(t)].$$

The right-hand side is a linear function of the parameter $p$, and linear least squares yields the value of $p$ that minimizes the sum of the squares of the residuals $I_{j+1} - I_j - pI_j(N - I_j)$.

- This doesn’t resolve the question of which value of $I(0)$ to use. If we let $t_0 = 0$ for the first data point, then we could let $I(0) = I_0$. However, this might not be the best choice of $I(0)$ in order to make the residuals $I_j - I(t_j)$ small.
Method 2 to use Linear Least Squares

- Going back to the solution of model (3), we can make a transformation of variables so that the transformed solution does depend linearly on its parameters. First we divide both sides into $N$ and simplify:

$$N/I(t) = 1 + [N/I(0) - 1]e^{-pNt}$$

- Next subtract 1 and take the logarithm:

$$\log[N/I(t) - 1] = \log[N/I(0) - 1] - pNt$$

- Let $Z(t) = \log[N/I(t) - 1]$; then the model becomes $Z(t) = Z(0) - pNt$. This is a linear function of the parameters $pN$ and $Z(0)$. One can transform the data to pairs $(t_j, Z_j)$, use linear least squares to determine values for $pN$ and $Z(0)$, and then solve for $p$ and $I(0)$. 
Caveat

• Both methods of using linear least squares transform the model or its solution into a linear relationship between two quantities that can be computed from the data points \((t_j, I_j)\); in the second method, the model predicts that \(Z_j\) is a linear function of \(t_j\).

• Rather than simply accept the result of the least squares fit, one should graph the predicted relationship (e.g., \(Z_j\) versus \(t_j\)) and see if it actually looks linear. This gives some idea of how valid the model is.

• Regardless of how one determines values for \(p\) and \(I(0)\), one should also check directly how well the resulting \(I(t)\) fits the data.