Representation and approximation of data

Wojciech Czaja and Brian Hunt

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Outline

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Frames

- Practical potential was not recognized until the 1990s.
- Among the generalizations of frames, many ideas have been proposed in the recent years, e.g., frames of subspaces (Casazza and Kutyniok), pseudo-frames (Li and Ogawa), fusion frames (Casazza, FIckus, Kutyniok), outer frames (Aldrubi, Cabrelli, and Molter), *g*-frames (Sun), and multiplicative frames (Benedetto).
- Frames have a simple interpretation in the context of fine dimensional vector spaces.

Finite frames

Definition

A collection $\{x_n\}_{n=1}^N$ in a Hilbert space $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist $0 < A \le B < \infty$ such that

$$\forall x \in \mathbb{H}, \; A\|x\|^2 \le \sum_{n=1}^{N} |\langle x, x_n\rangle|^2 \le B\|x\|^2.$$

The constants A and B are the *frame bounds*. If $A = B$, the frame is an *A-tight* frame.

- Any spanning set of vectors in $\mathbb{R}^d$ is a *frame* for $\mathbb{R}^d$.
- However, the spanning property does not indicate the value of frames for representation and stability in noisy environments.

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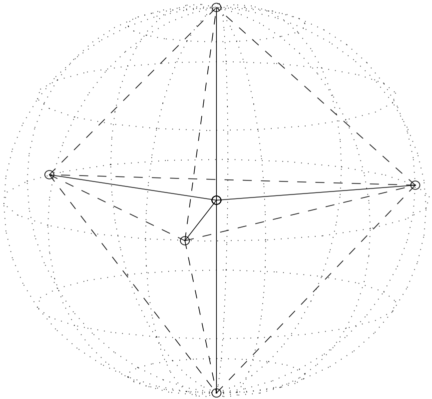
Example of a frame



Typical frames are redundant systems with more elements that the dimensionality of the space they represent.

From data to frame operators

- Given data space X of N vectors $x_n \in \mathbb{R}^D$. Without loss of generality, assume $\sum x_n = 0$ (subtract mean).
- Let P be $D \times N$ matrix whose columns are the data vectors x_n.
- Let $\mathbb{H} = \text{span}\{x_n\}_{n=1}^N \subseteq \mathbb{R}^D$. Define $L : \mathbb{H} \to \mathbb{R}^N$,

$$v \mapsto P^* v = L(v) = \{\langle v, x_n \rangle\} \in \mathbb{R}^N,$$

and its Hilbert space adjoint $L^* : \mathbb{R}^N \to \mathbb{H} \subseteq \mathbb{R}^D$,

$$w \mapsto L^*(w) = \sum_{n=1}^{N} w[n] x_n, \quad w = (w[1], w[2], \ldots, w[N]).$$

- L is the *Bessel (analysis)* operator, and L^* is the *synthesis* operator.

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Finite frames

- Recall the Bessel operator $L(v) = \{\langle v, x_n \rangle\} \in \mathbb{R}^N$.
- The *frame operator* for $\mathbb{H}$ is

$$S = L^* L : \mathbb{H} \to \mathbb{H}.$$

- $\{x_n\}_{n=1}^N$ is a *frame* for $\mathbb{H}$ if

$$\exists \; 0 < A \leq B < \infty \quad \text{such that} \quad AI \leq S \leq BI.$$

- $AI \leq S \leq BI$ implies that S is invertible and that $B^{-1} I \leq S \leq A^{-1} I$.

Finite frames

Theorem

a. $\{x_n\}_{n=1}^N$ *is a frame for* $\mathbb{H}$ *if and only if*

$$\forall\, v \in \mathbb{H}, v = \sum_{n=1}^{N} \langle v, S^{-1}(x_n)\rangle x_n = \sum_{n=1}^{N} \langle v, x_n\rangle S^{-1}(x_n).$$

b. $\{x_n\}_{n=1}^N$ *is an A-tight frame for* $\mathbb{H}$ *if and only if* $S = AI$.

P. Casazza, "The art of frame theory," arXiv preprint math/9910168, 1999.

Tight frame vs. ONB

Theorem (Vitali, 1921)

Let H be a Hilbert space, $\{x_n\} \subseteq H$, $\|x_n\| = 1$.

$$\{x_n\} \text{ is } 1\text{-tight} \Leftrightarrow \{x_n\} \text{ is an ONB.}$$

Proof. If $\{x_n\}$ is 1-tight, then $\forall y \in H$, $\|y\|^2 = \sum_n |\langle y, x_n \rangle|^2$. Since each $\|x_n\| = 1$, we have

$$1 = \|x_n\|^2 = \sum_k |\langle x_n, x_k \rangle|^2 = 1 + \sum_{k, k \neq n} |\langle x_n, x_k \rangle|^2$$

$$\Rightarrow \sum_{k \neq n} |\langle x_n, x_k \rangle|^2 = 0 \Rightarrow \forall n \neq k, \ \langle x_n, x_k \rangle = 0.$$

The role of covariance

- The frame operator S can be written as

$$S : \mathbb{H} \to \mathbb{H},\ v \mapsto \sum_{n=1}^{N} \langle v, x_n \rangle x_n = (PP^*)v,$$

  where PP^* is $D \times D$.

- Hence, up to a scaling factor and a translation, S is the linear operator identified with the $D \times D$ symmetric covariance matrix $C = \frac{1}{N} PP^*$ of the data space, i.e.

$$C = \frac{1}{N} \left(\sum_{j=1}^{N} x_j[m] x_j[n] \right)_{m,n=1}^{D}, \quad x_j = (x_j[1], \ldots, x_j[D]) \in \mathbb{R}^D.$$

The Grammian

- The *Grammian operator* for X is

$$G = LL^* : \mathbb{R}^N \to \mathbb{R}^N.$$

  Thus, G is $N \times N$ and

$$G = \{\langle x_m, x_n \rangle\}_{m,n=1}^N = P^* P.$$

- $G = LL^*$, $N \times N$, and $S = L^* L$, $D \times D$, have the same non-zero eigenvalues, a fact we shall exploit.

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FUNTFs

Let $\mathbb{K}$ be an r-dimensional Hilbert space, and let $\Psi_n \in \mathbb{K}$, $||\Psi_n|| = 1$, $n = 1, \ldots, s$.

- If $\{\Psi_n\}_{n=1}^{s}$ is a **finite unit norm tight frame (FUNTF)** for $\mathbb{K} = \mathbb{R}^r$, then

$$\forall \;\; y \in \mathbb{K}, \;\; y = \frac{s}{r} \sum_{n=1}^{s} \langle y, \Psi_n \rangle \Psi_n.$$

- *Problem*: Find FUNTFs analytically, effectively, and computationally.

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Characterization of FUNTFs

The total frame force potential energy is:

$$TFP(\{\Psi_n\}) = \sum_{m=1}^{s} \sum_{n=1}^{s} |\langle \Psi_m, \Psi_n \rangle|^2.$$

Theorem

- *Let $s = r$. The minimum value of TFP, for the frame force and s variables, is s; and the minimizers are precisely the orthonormal sets of s elements for $\mathbb{K}$.*
- *Let $s > r$. The minimum value of TFP, for the frame force and s variables, is s^2/r; and the minimizers are precisely the FUNTFs of s elements for $\mathbb{K}$.*

J. J. Benedetto and M. Fickus, "Finite normalized tight frames," Adv. Comp. Math., 2003, Vol. 18, pp. 357–385.

Examples of FUNTFs

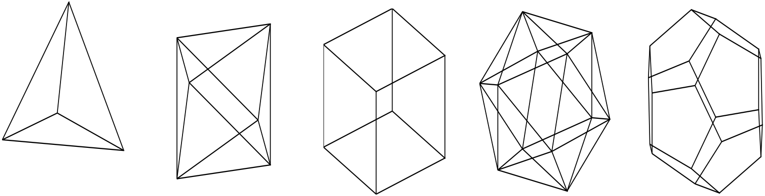

Figure : The vertices of the Platonic solids are examples of finite unit norm tight frames.

R. Vale, S. Waldron, “The vertices of the Platonic solids are tight frames,” in: Proceedings of the Conference on Advances in Constructive Approximation (M. Neamtu, E. B. Saff, eds.). Brentwood, TN: Nashboro Press, 2004.

Examples of FUNTFs

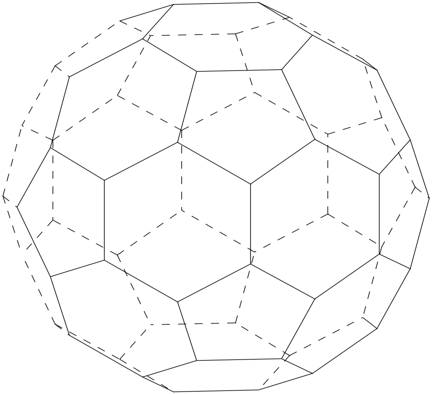

Figure : The truncated icosahedron (also known as the “soccer ball” or “bucky ball”) forms a tight frame for n-dimensional Euclidean space.

Motivation for frames

- Different classes of interest may not be orthogonal to each other; however, they may be captured by different frame elements. It is plausible that classes may correspond to elements in a frame but not elements in a basis.
- A *frame* generalizes the concept of an orthonormal basis. Frame elements are non–orthogonal.
- Frames provide over-complete data decompositions, often useful for numerical stability and noise reduction.