Representation and approximation of data

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Lecture 4: Overcomplete Representations
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Practical potential was not recognized until the 1990s.

Among the generalizations of frames, many ideas have been proposed in the recent years, e.g., frames of subspaces (Casazza and Kutyniok), pseudo-frames (Li and Ogawa), fusion frames (Casazza, Flckus, Kutyniok), outer frames (Aldrubi, Cabrelli, and Molter), $g$-frames (Sun), and multiplicative frames (Benedetto).

Frames have a simple interpretation in the context of fine dimensional vector spaces.
Finite frames

Definition

A collection \( \{x_n\}_{n=1}^N \) in a Hilbert space \( \mathbb{H} \) is a frame for \( \mathbb{H} \) if there exist \( 0 < A \leq B < \infty \) such that

\[
\forall x \in \mathbb{H}, \quad A\|x\|^2 \leq \sum_{n=1}^N |\langle x, x_n \rangle|^2 \leq B\|x\|^2.
\]

The constants \( A \) and \( B \) are the frame bounds. If \( A = B \), the frame is an \( A \)-tight frame.

- Any spanning set of vectors in \( \mathbb{R}^d \) is a frame for \( \mathbb{R}^d \).
- However, the spanning property does not indicate the value of frames for representation and stability in noisy environments.
Typical frames are redundant systems with more elements that the dimensionality of the space they represent.
Given data space $X$ of $N$ vectors $x_n \in \mathbb{R}^D$. Without loss of generality, assume $\sum x_n = 0$ (subtract mean).

Let $P$ be $D \times N$ matrix whose columns are the data vectors $x_n$.

Let $\mathcal{H} = \text{span}\{x_n\}_{n=1}^N \subseteq \mathbb{R}^D$. Define $L : \mathcal{H} \rightarrow \mathbb{R}^N$,

$$v \mapsto P^* v = L(v) = \{\langle v, x_n \rangle \} \in \mathbb{R}^N,$$

and its Hilbert space adjoint $L^* : \mathbb{R}^N \rightarrow \mathcal{H} \subseteq \mathbb{R}^D$,

$$w \mapsto L^*(w) = \sum_{n=1}^N w[n]x_n, \quad w = (w[1], w[2], \ldots, w[N]).$$

$L$ is the Bessel (analysis) operator, and $L^*$ is the synthesis operator.
Recall the Bessel operator $L(v) = \{\langle v, x_n \rangle \} \in \mathbb{R}^N$.

The frame operator for $\mathbb{H}$ is

$$S = L^* L : \mathbb{H} \to \mathbb{H}.$$ 

$\{x_n\}_{n=1}^N$ is a frame for $\mathbb{H}$ if

$$\exists \ 0 < A \leq B < \infty \ \text{such that} \ Al \leq S \leq Bl.$$ 

$Al \leq S \leq Bl$ implies that $S$ is invertible and that

$$B^{-1} I \leq S \leq A^{-1} I.$$
Finite frames

**Theorem**

**a.** \( \{x_n\}_{n=1}^N \) is a frame for \( \mathbb{H} \) if and only if

\[
\forall v \in \mathbb{H}, \quad v = \sum_{n=1}^{N} \langle v, S^{-1}(x_n) \rangle x_n = \sum_{n=1}^{N} \langle v, x_n \rangle S^{-1}(x_n).
\]

**b.** \( \{x_n\}_{n=1}^N \) is an A-tight frame for \( \mathbb{H} \) if and only if \( S = AI \).

Theorem (Vitali, 1921)

Let $H$ be a Hilbert space, $\{x_n\} \subseteq H$, $\|x_n\| = 1$.

\[
\{x_n\} \text{ is 1-tight } \iff \{x_n\} \text{ is an ONB}.
\]

Proof. If $\{x_n\}$ is 1-tight, then $\forall y \in H$, $\|y\|^2 = \sum_n |\langle y, x_n \rangle|^2$. Since each $\|x_n\| = 1$, we have

\[
1 = \|x_n\|^2 = \sum_k |\langle x_n, x_k \rangle|^2 = 1 + \sum_{k,k \neq n} |\langle x_n, x_k \rangle|^2
\]

\[
\Rightarrow \sum_{k \neq n} |\langle x_n, x_k \rangle|^2 = 0 \Rightarrow \forall n \neq k, \langle x_n, x_k \rangle = 0.
\]
The role of covariance

- The frame operator $S$ can be written as

$$S : \mathbb{H} \rightarrow \mathbb{H}, \ \nu \mapsto \sum_{n=1}^{N} \langle \nu, x_n \rangle x_n = (PP^*)\nu,$$

where $PP^*$ is $D \times D$.

- Hence, up to a scaling factor and a translation, $S$ is the linear operator identified with the $D \times D$ symmetric covariance matrix $C = \frac{1}{N} PP^*$ of the data space, i.e.

$$C = \frac{1}{N} \left( \sum_{j=1}^{N} x_j[m] x_j[n] \right)^D_{m,n=1}, \ \ x_j = (x_j[1], \ldots, x_j[D]) \in \mathbb{R}^D.$$
The Grammian operator for $X$ is

$$G = LL^* : \mathbb{R}^N \rightarrow \mathbb{R}^N.$$ 

Thus, $G$ is $N \times N$ and

$$G = \{ \langle x_m, x_n \rangle \}_{m,n=1}^{N} = P^* P.$$ 

$G = LL^*$, $N \times N$, and $S = L^* L$, $D \times D$, have the same non-zero eigenvalues, a fact we shall exploit.
Let $\mathbb{K}$ be an $r$-dimensional Hilbert space, and let $\psi_n \in \mathbb{K}$, $\|\psi_n\| = 1$, $n = 1, \ldots, s$.

- If $\{\psi_n\}_{n=1}^s$ is a finite unit norm tight frame (FUNTF) for $\mathbb{K} = \mathbb{R}^r$, then

$$\forall \ y \in \mathbb{K}, \ y = \frac{s}{r} \sum_{n=1}^{s} \langle y, \psi_n \rangle \psi_n.$$

- **Problem**: Find FUNTFs analytically, effectively, and computationally.
The total frame force potential energy is:

\[
TFP(\{\Psi_n\}) = \sum_{m=1}^{s} \sum_{n=1}^{s} |\langle \Psi_m, \Psi_n \rangle|^2.
\]

**Theorem**

- Let \( s = r \). The minimum value of TFP, for the frame force and \( s \) variables, is \( s \); and the minimizers are precisely the orthonormal sets of \( s \) elements for \( \mathbb{K} \).

- Let \( s > r \). The minimum value of TFP, for the frame force and \( s \) variables, is \( s^2/r \); and the minimizers are precisely the FUNTFs of \( s \) elements for \( \mathbb{K} \).

Examples of FUNTFs

Figure: The vertices of the Platonic solids are examples of finite unit norm tight frames.

Examples of FUNTFs

Figure: The truncated icosahedron (also known as the “soccer ball” or “bucky ball”) forms a tight frame for n-dimensional Euclidean space.
Motivation for frames

- Different classes of interest may not be orthogonal to each other; however, they may be captured by different frame elements. It is plausible that classes may correspond to elements in a frame but not elements in a basis.

- A frame generalizes the concept of an orthonormal basis. Frame elements are non–orthogonal.

- Frames provide over-complete data decompositions, often useful for numerical stability and noise reduction.