Modeling Portfolios that Contain Risky Assets

Optimization II: Model-Based Target Portfolios

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Optimization II: Model-Based Target Portfolios

We now address how to select a target portfolio that contains N risky assets along with a risk-free safe investment and possibly a risk-free credit line. Given the mean vector $\mathbf{m}$, the covariance matrix $\mathbf{V}$, and the risk-free rates μ_{si} and μ_{cl}, the idea is to select the portfolio distribution $\mathbf{f}$ that maximizes an objective function of the form

$$\widehat{\Gamma}(\mathbf{f}) = \widehat{\mu} - \tfrac{1}{2}\widehat{\sigma}^2 - \chi\,\widehat{\sigma}\,,$$

where

$$\widehat{\mu} = \mu_{\mathsf{rf}}\left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \mathbf{m}^{\mathsf{T}}\mathbf{f}\,, \qquad \widehat{\sigma} = \sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,, \qquad \mu_{\mathsf{rf}} = \begin{cases} \mu_{\mathsf{si}} & \text{for } \mathbf{1}^{\mathsf{T}}\mathbf{f} < 1\,, \\ \mu_{\mathsf{cl}} & \text{for } \mathbf{1}^{\mathsf{T}}\mathbf{f} > 1\,. \end{cases}$$

Here $\chi = \zeta/\sqrt{T}$ where $\zeta \geq 0$ is the *risk aversion coefficient* and $T > 0$ is a time horizon that is usually the *time to the next portfolio rebalancing*. Both ζ and T are chosen by the investor.

Reduced Maximization Problem. *Because frontier portfolios minimize $\widehat{\sigma}$ for a given value of $\widehat{\mu}$, the optimal $\mathbf{f}$ clearly must be a frontier portfolio. Because the optimal portfolio must also be more efficient than every other portfolio with the same volatility, it must lie on the efficient frontier.*

Recall that the efficient frontier is a curve $\mu = \mu_{\text{ef}}(\sigma)$ in the $\sigma\mu$-plane given by an increasing, concave, continuous, piecewise differentiable function $\mu_{\text{ef}}(\sigma)$ that is defined over $[0, \infty)$ for the unconstrained One Risk-Free Rate and Two Risk-Free Rates models, and over $[0, \sigma_{\text{mx}}]$ for the long portfolio model. The problem thereby reduces to finding σ that maximizes

$$\Gamma_{\text{ef}}(\sigma) = \mu_{\text{ef}}(\sigma) - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,.$$

This function has the piecewise derivative $\Gamma'_{\text{ef}}(\sigma) = \mu'_{\text{ef}}(\sigma) - \sigma - \chi$. Because $\mu_{\text{ef}}(\sigma)$ is concave, $\Gamma'_{\text{ef}}(\sigma)$ is strictly decreasing.

Because $\Gamma'_{\mathrm{ef}}(\sigma)$ is strictly decreasing, there are three possibilities.

- $\Gamma_{\mathrm{ef}}(\sigma)$ takes its maximum at $\sigma = 0$, the left endpoint of its interval of definition. This case arises whenever $\Gamma'_{\mathrm{ef}}(0) \leq 0$.

- $\Gamma_{\mathrm{ef}}(\sigma)$ takes its maximum in the interior of its interval of definition at the unique point $\sigma = \sigma_{\mathrm{opt}}$ where $\Gamma'_{\mathrm{ef}}(\sigma) = \mu'_{\mathrm{ef}}(\sigma) - \sigma - \chi$ changes sign. This case arises for the unconstrained models whenever $\Gamma'_{\mathrm{ef}}(0) > 0$, and for the long portfolio model whenever $\Gamma'_{\mathrm{ef}}(\sigma_{\mathrm{mx}}) < 0 < \Gamma'_{\mathrm{ef}}(0)$.

- $\Gamma_{\mathrm{ef}}(\sigma)$ takes its maximum at $\sigma = \sigma_{\mathrm{mx}}$, the right endpoint of its interval of definition. This case arises only for the long portfolio model whenever $\Gamma'_{\mathrm{ef}}(\sigma_{\mathrm{mx}}) \geq 0$.

This reduced maximization problem can be visualized by considering the family of parabolas parameterized by Γ as

$$\mu = \Gamma + \chi\sigma + \tfrac{1}{2}\sigma^2 \,.$$

As Γ varies the graph of this parabola shifts up and down in the $\sigma\mu$-plane. For some values of Γ the corresponding parabola will intersect the efficient frontier, which is given by $\mu = \mu_{\mathsf{ef}}(\sigma)$. There is clearly a maximum such Γ. As the parabola is strictly convex while the efficient frontier is concave, for this maximum Γ the intersection will consist of a single point $(\sigma_{\mathsf{opt}}, \mu_{\mathsf{opt}})$. Then $\sigma = \sigma_{\mathsf{opt}}$ is the maximizer of $\Gamma_{\mathsf{ef}}(\sigma)$.

This reduction is appealing because the efficient frontier only depends on general information about an investor, like whether he or she will take short positions. Once it is computed, the problem of maximizing any given $\widehat{\Gamma}(\mathbf{f})$ over all admissible portfolios $\mathbf{f}$ reduces to the problem of maximizing the associated $\Gamma_{\mathsf{ef}}(\sigma)$ over all admissible σ — a problem over one variable.

In summary, our approach to portfolio selection has three steps:

1. Choose a return rate history over a given period (say the past year) and calibrate the mean vector $\mathbf{m}$ and the covariance matrix $\mathbf{V}$ with it.

2. Given $\mathbf{m}$, $\mathbf{V}$, μ_{si}, μ_{cl}, and any portfolio constraints, compute $\mu_{\mathsf{ef}}(\sigma)$.

3. Finally, choose $\chi = \zeta/\sqrt{T}$ and maximize the associated $\Gamma_{\mathsf{ef}}(\sigma)$; the maximizer σ_{opt} corresponds to a unique efficient frontier portfolio.

Below we will illustrate the last step on some models we have developed.

One Risk-Free Rate Model. This is the easiest model to analyze. You first compute σ_{mv}, μ_{mv}, and ν_{as} from the return rate history. The model assumes that $\mu_{\text{si}} = \mu_{\text{cl}} < \mu_{\text{mv}}$. Then its tangency parameters are

$$\nu_{\text{tg}} = \nu_{\text{as}} \sqrt{1 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{rf}}}{\nu_{\text{as}}\, \sigma_{\text{mv}}} \right)^2}\,, \qquad \sigma_{\text{tg}} = \sigma_{\text{mv}} \sqrt{1 + \left(\frac{\nu_{\text{as}}\, \sigma_{\text{mv}}}{\mu_{\text{mv}} - \mu_{\text{rf}}} \right)^2}\,,$$

where $\mu_{\text{rf}} = \mu_{\text{si}} = \mu_{\text{cl}}$, while its efficient frontier is

$$\mu_{\text{ef}}(\sigma) = \mu_{\text{rf}} + \nu_{\text{tg}}\, \sigma \qquad \text{for } \sigma \in [0, \infty)\,.$$

Because $\Gamma_{\text{ef}}(\sigma) = \mu_{\text{ef}}(\sigma) - \frac{1}{2}\sigma^2 - \chi\sigma$, we have

$$\Gamma'_{\text{ef}}(\sigma) = \nu_{\text{tg}} - \sigma - \chi\,.$$

When $\chi \geq \nu_{\text{tg}}$ we see that $\Gamma'_{\text{ef}}(0) = \nu_{\text{tg}} - \chi \leq 0$, whereby $\sigma_{\text{opt}} = 0$, while when $\chi < \nu_{\text{tg}}$ there is a positive solution of $\Gamma'_{\text{ef}}(\sigma) = 0$. We obtain

$$\sigma_{\text{opt}} = \begin{cases} 0 & \text{if } \nu_{\text{tg}} \leq \chi\,, \\ \nu_{\text{tg}} - \chi & \text{if } \chi < \nu_{\text{tg}}\,. \end{cases}$$

The optimal return rate $\mu_{\mathsf{opt}} = \mu_{\mathsf{ef}}(\sigma_{\mathsf{opt}})$ is expressed in terms of the return rate μ_{tg} of the tangency portfolio and the risk-free rate μ_{rf} as

$$\mu_{\mathsf{opt}} = \left(1 - \frac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{tg}}}\right)\mu_{\mathsf{rf}} + \frac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{tg}}}\,\mu_{\mathsf{tg}}\,,$$

where

$$\mu_{\mathsf{tg}} = \mu_{\mathsf{mv}} + \frac{\nu_{\mathsf{as}}^2\,\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}\,.$$

The optimal efficient frontier portfolio has the distribution $\mathbf{f}_{\mathsf{opt}} = \mathbf{f}_{\mathsf{ef}}(\sigma_{\mathsf{opt}})$ which is expressed in terms of the tangency portfolio $\mathbf{f}_{\mathsf{tg}}$ as

$$\mathbf{f}_{\mathsf{opt}} = \frac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{tg}}}\,\mathbf{f}_{\mathsf{tg}}\,, \qquad \text{where} \quad \mathbf{f}_{\mathsf{tg}} = \frac{\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}\,\mathbf{V}^{-1}\big(\mathbf{m} - \mu_{\mathsf{rf}}\mathbf{1}\big)\,.$$

It follows from the distribution $\mathbf{f}_{\mathrm{opt}}$ that the optimal efficient frontier portfolio can be built from the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$ and the risk-free assets as follows. There are four possibilities:

1. If $\sigma_{\mathrm{opt}} = 0$ then the investor will hold only the safe investment.

2. If $\sigma_{\mathrm{opt}} \in (0, \sigma_{\mathrm{tg}})$ then the investor will place

$$\frac{\sigma_{\mathrm{tg}} - \sigma_{\mathrm{opt}}}{\sigma_{\mathrm{tg}}} \quad \text{of the portfolio value in the safe investment} \,,$$
$$\frac{\sigma_{\mathrm{opt}}}{\sigma_{\mathrm{tg}}} \quad \text{of the portfolio value in the tangency portfolio } \mathbf{f}_{\mathrm{tg}} \,.$$

3. If $\sigma_{\mathrm{opt}} = \sigma_{\mathrm{tg}}$ then the investor will hold only the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$.

4. If $\sigma_{\mathrm{opt}} \in (\sigma_{\mathrm{tg}}, \infty)$ then the investor will place

$$\frac{\sigma_{\mathrm{opt}}}{\sigma_{\mathrm{tg}}} \quad \text{of the portfolio value in the tangency portfolio } \mathbf{f}_{\mathrm{tg}} \,,$$

$$\text{by borrowing} \quad \frac{\sigma_{\mathrm{opt}} - \sigma_{\mathrm{tg}}}{\sigma_{\mathrm{tg}}} \quad \text{of this value from the credit line} \,.$$

In order to see which of these four cases arises as a function of μ_{rf}, we must compare ν_{tg} with χ and $\sigma_{\mathrm{opt}} = \nu_{\mathrm{tg}} - \chi$ with σ_{tg}. Because $\nu_{\mathrm{tg}} > \nu_{\mathrm{as}}$, the condition $\chi \geq \nu_{\mathrm{tg}}$ cannot be met unless $\chi > \nu_{\mathrm{as}}$, in which case it can be expressed as $\mu_{\mathrm{rf}} \geq \eta_{\mathrm{ex}}(\chi)$ where

$$\eta_{\mathrm{ex}}(\chi) = \begin{cases} \mu_{\mathrm{mv}} & \text{if } \chi \leq \nu_{\mathrm{as}} \,, \\ \mu_{\mathrm{mv}} - \sigma_{\mathrm{mv}} \sqrt{\chi^2 - \nu_{\mathrm{as}}^2} & \text{if } \chi > \nu_{\mathrm{as}} \,. \end{cases}$$

This is called the *exit rate* for the investor because when $\mu_{\mathrm{rf}} \geq \eta_{\mathrm{ex}}(\chi)$ the investor is likely to sell all of his or her risky assets.

In order to compare $\nu_{\mathsf{tg}} - \chi$ with σ_{tg}, notice that $\nu_{\mathsf{tg}} - \chi = \sigma_{\mathsf{tg}}$ whenever $\mu = \mu_{\mathsf{rf}}$ solves the equation

$$\nu_{\mathsf{as}}\sqrt{1 + \left(\frac{\mu_{\mathsf{mv}} - \mu}{\nu_{\mathsf{as}}\,\sigma_{\mathsf{mv}}}\right)^2} - \sigma_{\mathsf{mv}}\sqrt{1 + \left(\frac{\nu_{\mathsf{as}}\,\sigma_{\mathsf{mv}}}{\mu_{\mathsf{mv}} - \mu}\right)^2} = \chi\,.$$

The left-hand side of this equation is a strictly decreasing function of μ over the interval $(-\infty, \mu_{\mathsf{mv}})$ that maps onto $\mathbb{R}$. Let $\mu = \eta_{\mathsf{tg}}(\chi)$ be the unique solution of this equation in $(-\infty, \mu_{\mathsf{mv}})$. Then because $\sigma_{\mathsf{opt}} = \nu_{\mathsf{tg}} - \chi$, we see that the four cases arise as follows

$$\begin{aligned}
&\sigma_{\mathsf{opt}} = 0 && \text{if and only if} && \eta_{\mathsf{ex}}(\chi) \leq \mu_{\mathsf{rf}}\,,\\
&\sigma_{\mathsf{opt}} \in (0, \sigma_{\mathsf{tg}}) && \text{if and only if} && \eta_{\mathsf{tg}}(\chi) < \mu_{\mathsf{rf}} < \eta_{\mathsf{ex}}(\chi)\,,\\
&\sigma_{\mathsf{opt}} = \sigma_{\mathsf{tg}} && \text{if and only if} && \mu_{\mathsf{rf}} = \eta_{\mathsf{tg}}(\chi)\,,\\
&\sigma_{\mathsf{opt}} \in (\sigma_{\mathsf{tg}}, \infty) && \text{if and only if} && \mu_{\mathsf{rf}} < \eta_{\mathsf{tg}}(\chi)\,.
\end{aligned}$$

Notice that if $\chi \leq \nu_{\mathsf{as}}$ then $\eta_{\mathsf{ex}}(\chi) = \mu_{\mathsf{mv}}$, so the first case does not arise.

In particular, when $\chi = 0$ the first case does not arise, while $\mu = \eta_{\mathsf{tg}}(0)$ is the solution of

$$\nu_{\mathsf{as}}\sqrt{1 + \left(\frac{\mu_{\mathsf{mv}} - \mu}{\nu_{\mathsf{as}}\,\sigma_{\mathsf{mv}}}\right)^2} - \sigma_{\mathsf{mv}}\sqrt{1 + \left(\frac{\nu_{\mathsf{as}}\,\sigma_{\mathsf{mv}}}{\mu_{\mathsf{mv}} - \mu}\right)^2} = 0\,.$$

This can be solved explicitly to find that

$$\eta_{\mathsf{tg}}(0) = \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2\,.$$

Therefore the three remaining cases arise as follows:

$$\begin{aligned}
&\sigma_{\mathsf{opt}} \in (0, \sigma_{\mathsf{tg}}) &&\text{if and only if} &&\mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2 < \mu_{\mathsf{rf}}\,,\\
&\sigma_{\mathsf{opt}} = \sigma_{\mathsf{tg}} &&\text{if and only if} &&\mu_{\mathsf{rf}} = \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2\,,\\
&\sigma_{\mathsf{opt}} \in (\sigma_{\mathsf{tg}}, \infty) &&\text{if and only if} &&\mu_{\mathsf{rf}} < \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2\,.
\end{aligned}$$

Specifically, when $\chi = 0$ we have

$$\sigma_{\mathsf{opt}} = \nu_{\mathsf{tg}}\,, \qquad \mu_{\mathsf{opt}} = \mu_{\mathsf{rf}} + \nu_{\mathsf{tg}}^2\,, \qquad \gamma_{\mathsf{opt}} = \mu_{\mathsf{rf}} + \tfrac{1}{2}\nu_{\mathsf{tg}}^2\,.$$

Two Risk-Free Rates Model. This is the next easiest model to analyze. We first compute σ_{mv}, μ_{mv}, and ν_{as} from the return rate history. The model assumes that $\mu_{\text{si}} < \mu_{\text{cl}} < \mu_{\text{mv}}$. Then its tangency parameters are

$$\nu_{\text{st}} = \nu_{\text{as}} \sqrt{1 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{si}}}{\nu_{\text{as}}\, \sigma_{\text{mv}}} \right)^2}, \qquad \sigma_{\text{st}} = \sigma_{\text{mv}} \sqrt{1 + \left(\frac{\nu_{\text{as}}\, \sigma_{\text{mv}}}{\mu_{\text{mv}} - \mu_{\text{si}}} \right)^2},$$

$$\nu_{\text{ct}} = \nu_{\text{as}} \sqrt{1 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{cl}}}{\nu_{\text{as}}\, \sigma_{\text{mv}}} \right)^2}, \qquad \sigma_{\text{ct}} = \sigma_{\text{mv}} \sqrt{1 + \left(\frac{\nu_{\text{as}}\, \sigma_{\text{mv}}}{\mu_{\text{mv}} - \mu_{\text{cl}}} \right)^2},$$

while its efficient frontier is

$$\mu_{\text{ef}}(\sigma) = \begin{cases} \mu_{\text{si}} + \nu_{\text{st}}\, \sigma & \text{for } \sigma \in [0, \sigma_{\text{st}}]\,, \\ \mu_{\text{mv}} + \nu_{\text{as}} \sqrt{\sigma^2 - \sigma_{\text{mv}}^2} & \text{for } \sigma \in [\sigma_{\text{st}}, \sigma_{\text{ct}}]\,, \\ \mu_{\text{cl}} + \nu_{\text{ct}}\, \sigma & \text{for } \sigma \in [\sigma_{\text{ct}}, \infty)\,. \end{cases}$$

Because $\Gamma_{\text{ef}}(\sigma) = \mu_{\text{ef}}(\sigma) - \frac{1}{2}\sigma^2 - \chi\sigma$, we have

$$\Gamma'_{\text{ef}}(\sigma) = \begin{cases} \nu_{\text{st}} - \sigma - \chi & \text{for } \sigma \in [0, \sigma_{\text{st}}]\,, \\ \dfrac{\nu_{\text{as}}\,\sigma}{\sqrt{\sigma^2 - \sigma_{\text{mv}}^2}} - \sigma - \chi & \text{for } \sigma \in [\sigma_{\text{st}}, \sigma_{\text{ct}}]\,, \\ \nu_{\text{ct}} - \sigma - \chi & \text{for } \sigma \in [\sigma_{\text{ct}}, \infty)\,. \end{cases}$$

When $\chi \geq \nu_{\text{tg}}$ we see that $\Gamma'_{\text{ef}}(0) = \nu_{\text{tg}} - \chi \leq 0$, whereby $\sigma_{\text{opt}} = 0$, while when $\chi < \nu_{\text{tg}}$ there is a positive solution of $\Gamma'_{\text{ef}}(\sigma) = 0$. We obtain

$$\sigma_{\text{opt}} = \begin{cases} 0 & \text{if } \nu_{\text{st}} \leq \chi\,, \\ \nu_{\text{st}} - \chi & \text{if } \nu_{\text{st}} - \sigma_{\text{st}} \leq \chi < \nu_{\text{st}}\,, \\ \sigma_{\text{q}}(\chi) & \text{if } \nu_{\text{ct}} - \sigma_{\text{ct}} \leq \chi < \nu_{\text{st}} - \sigma_{\text{st}}\,, \\ \nu_{\text{ct}} - \chi & \text{if } \chi < \nu_{\text{ct}} - \sigma_{\text{ct}}\,, \end{cases}$$

where $\sigma = \sigma_{\text{q}}(\chi) \in [\sigma_{\text{st}}, \sigma_{\text{ct}}]$ solves the quartic equation

$$\nu_{\text{as}}^2\,\sigma^2 = \left(\sigma^2 - \sigma_{\text{mv}}^2\right)(\sigma + \chi)^2\,.$$

The optimal return rate $\mu_{\mathsf{opt}} = \mu_{\mathsf{ef}}(\sigma_{\mathsf{opt}})$ is expressed in terms of the return rate μ_{st} of the safe tangency portfolio, the return rate μ_{ct} of the credit tangency portfolio, and the risk-free rates μ_{si} and μ_{cl} as

$$\mu_{\mathsf{opt}} = \begin{cases} \left(1 - \dfrac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{st}}}\right)\mu_{\mathsf{si}} + \dfrac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{st}}}\,\mu_{\mathsf{st}} & \text{for } \sigma_{\mathsf{opt}} \in [0, \sigma_{\mathsf{st}}]\,, \\ \mu_{\mathsf{mv}} + \nu_{\mathsf{as}}\sqrt{\sigma_{\mathsf{opt}}^2 - \sigma_{\mathsf{mv}}^2} & \text{for } \sigma_{\mathsf{opt}} \in (\sigma_{\mathsf{st}}, \sigma_{\mathsf{ct}})\,, \\ \left(1 - \dfrac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{ct}}}\right)\mu_{\mathsf{cl}} + \dfrac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{ct}}}\,\mu_{\mathsf{ct}} & \text{for } \sigma_{\mathsf{opt}} \in [\sigma_{\mathsf{ct}}, \infty)\,, \end{cases}$$

where

$$\mu_{\mathsf{st}} = \mu_{\mathsf{mv}} + \frac{\nu_{\mathsf{as}}^2\,\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{si}}}\,, \qquad \mu_{\mathsf{ct}} = \mu_{\mathsf{mv}} + \frac{\nu_{\mathsf{as}}^2\,\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{cl}}}\,.$$

The optimal efficient frontier portfolio has the distribution $\mathbf{f}_{\mathsf{opt}} = \mathbf{f}_{\mathsf{ef}}(\sigma_{\mathsf{opt}})$ which is expressed in terms of the safe tangency portfolio $\mathbf{f}_{\mathsf{st}}$ and the credit tangency portfolio $\mathbf{f}_{\mathsf{ct}}$ as

$$\mathbf{f}_{\mathsf{opt}} = \begin{cases} \dfrac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{st}}}\,\mathbf{f}_{\mathsf{st}} & \text{for } \sigma_{\mathsf{opt}} \in [0, \sigma_{\mathsf{st}}]\,, \\ \dfrac{\mu_{\mathsf{ct}} - \mu_{\mathsf{opt}}}{\mu_{\mathsf{ct}} - \mu_{\mathsf{st}}}\,\mathbf{f}_{\mathsf{st}} + \dfrac{\mu_{\mathsf{opt}} - \mu_{\mathsf{st}}}{\mu_{\mathsf{ct}} - \mu_{\mathsf{st}}}\,\mathbf{f}_{\mathsf{ct}} & \text{for } \sigma_{\mathsf{opt}} \in (\sigma_{\mathsf{st}}, \sigma_{\mathsf{ct}})\,, \\ \dfrac{\sigma_{\mathsf{opt}}}{\sigma_{\mathsf{ct}}}\,\mathbf{f}_{\mathsf{ct}} & \text{for } \sigma_{\mathsf{opt}} \in [\sigma_{\mathsf{ct}}, \infty)\,, \end{cases}$$

where

$$\mathbf{f}_{\mathsf{st}} = \frac{\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{si}}}\,\mathbf{V}^{-1}\Big(\mathbf{m} - \mu_{\mathsf{si}}\mathbf{1}\Big)\,, \quad \mathbf{f}_{\mathsf{ct}} = \frac{\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{cl}}}\,\mathbf{V}^{-1}\Big(\mathbf{m} - \mu_{\mathsf{cl}}\mathbf{1}\Big)\,.$$

The optimal efficient frontier portfolio is constructed from the safe tangency portfolio $\mathbf{f}_{\mathrm{st}}$, the credit tangency portfolio $\mathbf{f}_{\mathrm{ct}}$, and the risk-free assets as follows. There are six possibilities:

1. If $\sigma_{\mathrm{opt}} = 0$ then the investor will hold only the safe investment.

2. If $\sigma_{\mathrm{opt}} \in (0, \sigma_{\mathrm{st}})$ then the investor places

   $\dfrac{\sigma_{\mathrm{st}} - \sigma_{\mathrm{opt}}}{\sigma_{\mathrm{st}}}$ of the portfolio value in the safe investment,

   $\dfrac{\sigma_{\mathrm{opt}}}{\sigma_{\mathrm{st}}}$ of the portfolio value in the safe tangency portfolio $\mathbf{f}_{\mathrm{st}}$.

3. If $\sigma_{\mathrm{opt}} = \sigma_{\mathrm{st}}$ then the investor holds only the safe tangency portfolio $\mathbf{f}_{\mathrm{st}}$.

4. If $\sigma_{\text{opt}} \in (\sigma_{\text{st}}, \sigma_{\text{ct}})$ then the investor places

$\dfrac{\mu_{\text{ct}} - \mu_{\text{opt}}}{\mu_{\text{ct}} - \mu_{\text{st}}}$ of the portfolio value in the safe tangency portfolio $\mathbf{f}_{\text{st}}$,

$\dfrac{\mu_{\text{opt}} - \mu_{\text{st}}}{\mu_{\text{ct}} - \mu_{\text{st}}}$ of the portfolio value in the credit tangency portfolio $\mathbf{f}_{\text{ct}}$.

5. If $\sigma_{\text{opt}} = \sigma_{\text{ct}}$ then the investor holds the credit tangency portfolio $\mathbf{f}_{\text{ct}}$.

6. If $\sigma_{\text{opt}} \in (\sigma_{\text{ct}}, \infty)$ then the investor places

$\dfrac{\sigma_{\text{opt}}}{\sigma_{\text{ct}}}$ of the portfolio value in the credit tangency portfolio $\mathbf{f}_{\text{ct}}$,

by borrowing $\dfrac{\sigma_{\text{opt}} - \sigma_{\text{ct}}}{\sigma_{\text{ct}}}$ of this value from the credit line.

Because $\eta_{\text{ex}}(\chi)$ and $\eta_{\text{tg}}(\chi)$ where defined so that

$$
\begin{aligned}
\chi &= \nu_{\text{st}} && \text{if and only if} && \eta_{\text{ex}}(\chi) = \mu_{\text{si}}\,, \\
\chi &= \nu_{\text{st}} - \sigma_{\text{st}} && \text{if and only if} && \eta_{\text{tg}}(\chi) = \mu_{\text{si}}\,, \\
\chi &= \nu_{\text{ct}} - \sigma_{\text{ct}} && \text{if and only if} && \eta_{\text{tg}}(\chi) = \mu_{\text{cl}}\,,
\end{aligned}
$$

the six cases arise as a function of μ_{si} and μ_{cl} as follows:

$$
\begin{aligned}
\sigma_{\text{opt}} &= 0 && \text{if and only if} && \eta_{\text{ex}}(\chi) \leq \mu_{\text{si}} < \mu_{\text{cl}}\,, \\
\sigma_{\text{opt}} &\in (0, \sigma_{\text{st}}) && \text{if and only if} && \eta_{\text{tg}}(\chi) < \mu_{\text{si}} < \eta_{\text{ex}}(\chi)\,, \\
\sigma_{\text{opt}} &= \sigma_{\text{st}} && \text{if and only if} && \mu_{\text{si}} = \eta_{\text{tg}}(\chi) < \mu_{\text{cl}}\,, \\
\sigma_{\text{opt}} &\in (\sigma_{\text{st}}, \sigma_{\text{ct}}) && \text{if and only if} && \mu_{\text{si}} < \eta_{\text{tg}}(\chi) < \mu_{\text{cl}}\,, \\
\sigma_{\text{opt}} &= \sigma_{\text{ct}} && \text{if and only if} && \mu_{\text{si}} < \mu_{\text{cl}} = \eta_{\text{tg}}(\chi)\,, \\
\sigma_{\text{opt}} &\in (\sigma_{\text{ct}}, \infty) && \text{if and only if} && \mu_{\text{si}} < \mu_{\text{cl}} < \eta_{\text{tg}}(\chi)\,.
\end{aligned}
$$

Notice that if $\chi \leq \nu_{\text{as}}$ then $\eta_{\text{ex}}(\chi) = \mu_{\text{mv}}$, so the first case does not arise.

In particular, when $\chi = 0$ the first case does not arise. Because

$$\eta_{\mathrm{tg}}(0) = \mu_{\mathrm{mv}} - \sigma_{\mathrm{mv}}^2\,,$$

the five remaining cases arise as follows:

$$\begin{aligned}
&\sigma_{\mathrm{opt}} \in (0, \sigma_{\mathrm{st}}) && \text{if and only if} && \mu_{\mathrm{mv}} - \sigma_{\mathrm{mv}}^2 < \mu_{\mathrm{si}} < \mu_{\mathrm{cl}}\,,\\
&\sigma_{\mathrm{opt}} = \sigma_{\mathrm{st}} && \text{if and only if} && \mu_{\mathrm{si}} = \mu_{\mathrm{mv}} - \sigma_{\mathrm{mv}}^2 < \mu_{\mathrm{cl}}\,,\\
&\sigma_{\mathrm{opt}} \in (\sigma_{\mathrm{st}}, \sigma_{\mathrm{ct}}) && \text{if and only if} && \mu_{\mathrm{si}} < \mu_{\mathrm{mv}} - \sigma_{\mathrm{mv}}^2 < \mu_{\mathrm{cl}}\,,\\
&\sigma_{\mathrm{opt}} = \sigma_{\mathrm{ct}} && \text{if and only if} && \mu_{\mathrm{si}} < \mu_{\mathrm{cl}} = \mu_{\mathrm{mv}} - \sigma_{\mathrm{mv}}^2\,,\\
&\sigma_{\mathrm{opt}} \in (\sigma_{\mathrm{ct}}, \infty) && \text{if and only if} && \mu_{\mathrm{si}} < \mu_{\mathrm{cl}} < \mu_{\mathrm{mv}} - \sigma_{\mathrm{mv}}^2\,.
\end{aligned}$$

Moreover, because $\sigma = \sigma_{\mathrm{q}}(0)$ is the solution of

$$\nu_{\mathrm{as}}^2 \sigma^2 = \left(\sigma^2 - \sigma_{\mathrm{mv}}^2\right) \sigma^2\,,$$

we find that

$$\sigma_{\mathrm{q}}(0) = \sqrt{\sigma_{\mathrm{mv}}^2 + \nu_{\mathrm{as}}^2}\,.$$

Specifically, when $\chi = 0$ we have

$$\sigma_{\mathsf{opt}} = \begin{cases} \nu_{\mathsf{st}} & \text{for } \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2 \leq \mu_{\mathsf{si}}\,, \\ \sqrt{\sigma_{\mathsf{mv}}^2 + \nu_{\mathsf{as}}^2} & \text{for } \mu_{\mathsf{si}} < \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2 < \mu_{\mathsf{cl}}\,, \\ \nu_{\mathsf{ct}} & \text{for } \mu_{\mathsf{cl}} \leq \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2\,, \end{cases}$$

$$\mu_{\mathsf{opt}} = \begin{cases} \mu_{\mathsf{si}} + \nu_{\mathsf{st}}^2 & \text{for } \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2 \leq \mu_{\mathsf{si}}\,, \\ \mu_{\mathsf{mv}} + \nu_{\mathsf{as}}^2 & \text{for } \mu_{\mathsf{si}} < \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2 < \mu_{\mathsf{cl}}\,, \\ \mu_{\mathsf{cl}} + \nu_{\mathsf{ct}}^2 & \text{for } \mu_{\mathsf{cl}} \leq \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2\,, \end{cases}$$

$$\gamma_{\mathsf{opt}} = \begin{cases} \mu_{\mathsf{si}} + \frac{1}{2}\nu_{\mathsf{st}}^2 & \text{for } \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2 \leq \mu_{\mathsf{si}}\,, \\ \gamma_{\mathsf{mv}} + \frac{1}{2}\nu_{\mathsf{as}}^2 & \text{for } \mu_{\mathsf{si}} < \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2 < \mu_{\mathsf{cl}}\,, \\ \mu_{\mathsf{cl}} + \frac{1}{2}\nu_{\mathsf{ct}}^2 & \text{for } \mu_{\mathsf{cl}} \leq \mu_{\mathsf{mv}} - \sigma_{\mathsf{mv}}^2\,, \end{cases}$$

where $\gamma_{\mathsf{mv}} = \mu_{\mathsf{mv}} - \frac{1}{2}\sigma_{\mathsf{mv}}^2$ is the expected growth rate of the minimum volatility portfolio.

Long Portfolio Model. This is the most complicated model that we will analyze. We first compute σ_{mv}, μ_{mv}, and ν_{as} from the return rate history.

We then construct the efficient branch of the long frontier. We saw how to do this by an iterative construction whenever $\mathbf{f}_{\mathsf{f}}(\overline{\mu}_0) \geq \mathbf{0}$ for some $\overline{\mu}_0$. Here we will assume that $\mathbf{f}_{\mathsf{mv}} \geq \mathbf{0}$ and set $\overline{\mu}_0 = \mu_{\mathsf{mv}}$. In that case we found that $\sigma_{\mathsf{lf}}(\mu)$ is a piecewise differentiable function over $[\mu_{\mathsf{mv}}, \mu_{\mathsf{mx}}]$ that is given by a list in the form

$$\sigma_{\mathsf{lf}}(\mu) = \sigma_{\bar{\mathsf{f}}_k}(\mu) \equiv \sqrt{\sigma_{\mathsf{mv}_k}^2 + \left(\frac{\mu - \mu_{\mathsf{mv}_k}}{\nu_{\mathsf{as}_k}}\right)^2} \qquad \text{for } \mu \in [\overline{\mu}_k, \overline{\mu}_{k+1}]\,,$$

where σ_{mv_k}, μ_{mv_k}, and ν_{as_k} are the frontier parameters associated with the vector $\overline{\mathbf{m}}_k$ and matrix $\overline{\mathbf{V}}_k$ that determined $\sigma_{\bar{\mathsf{f}}_k}(\mu)$ in the k^{th} step of our construction. In particular, $\sigma_{\mathsf{mv}_0} = \sigma_{\mathsf{mv}}$, $\mu_{\mathsf{mv}_0} = \mu_{\mathsf{mv}}$, and $\nu_{\mathsf{as}_0} = \nu_{\mathsf{as}}$ because $\overline{\mathbf{m}}_0 = \mathbf{m}$ and $\overline{\mathbf{V}}_0 = \mathbf{V}$.

Next, we construct the continuously differentiable function $\mu_{\mathsf{ef}}(\sigma)$ over $[0, \sigma_{\mathsf{mx}}]$ that determines the efficient frontier given the return rate μ_{si} of the safe investment. The form of this construction depends upon the tangent line to the curve $\sigma = \sigma_{\mathsf{lf}}(\mu)$ at the point $(\sigma_{\mathsf{mx}}, \mu_{\mathsf{mx}})$. This tangent line has μ-intercept η_{mx} and slope ν_{mx} given by

$$\eta_{\mathsf{mx}} = \mu_{\mathsf{mx}} - \frac{\sigma_{\mathsf{lf}}(\mu_{\mathsf{mx}})}{\sigma'_{\mathsf{lf}}(\mu_{\mathsf{mx}})}, \qquad \nu_{\mathsf{mx}} = \frac{1}{\sigma'_{\mathsf{lf}}(\mu_{\mathsf{mx}})}.$$

These parameters are related by

$$\nu_{\mathsf{mx}} = \frac{\mu_{\mathsf{mx}} - \eta_{\mathsf{mx}}}{\sigma_{\mathsf{mx}}}.$$

The cases $\mu_{\mathsf{si}} \geq \eta_{\mathsf{mx}}$ and $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$ are considered separately.

Case $\mu_{\mathrm{si}} \geq \eta_{\mathrm{mx}}$. Here the efficient long frontier is simply determined by

$$\mu_{\mathrm{ef}}(\sigma) = \mu_{\mathrm{si}} + \nu_{\mathrm{ef}}\,\sigma \qquad \text{for } \sigma \in [0, \sigma_{\mathrm{mx}}]\,,$$

where the slope of this linear function is given by

$$\nu_{\mathrm{ef}} = \frac{\mu_{\mathrm{mx}} - \mu_{\mathrm{si}}}{\sigma_{\mathrm{mx}}}\,.$$

Notice that $\mu_{\mathrm{si}} \geq \eta_{\mathrm{mx}}$ if and only if $\nu_{\mathrm{ef}} \leq \nu_{\mathrm{mx}}$.

Because $\Gamma_{\mathrm{ef}}(\sigma) = \mu_{\mathrm{ef}}(\sigma) - \frac{1}{2}\sigma^2 - \chi\sigma$,

$$\Gamma'_{\mathrm{ef}}(\sigma) = \nu_{\mathrm{ef}} - \sigma - \chi \quad \text{for } \sigma \in [0, \sigma_{\mathrm{mx}}]\,.$$

We therefore find that

$$\sigma_{\mathrm{opt}} = \begin{cases} 0 & \text{if } \nu_{\mathrm{ef}} \leq \chi\,, \\ \nu_{\mathrm{ef}} - \chi & \text{if } \nu_{\mathrm{ef}} - \sigma_{\mathrm{mx}} \leq \chi < \nu_{\mathrm{ef}}\,, \\ \sigma_{\mathrm{mx}} & \text{if } \chi < \nu_{\mathrm{ef}} - \sigma_{\mathrm{mx}}\,. \end{cases}$$

Case $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$. In this case there is a tangent line with μ-intercept μ_{si}. The tangent line to the long frontier at the point $(\overline{\sigma}_k, \overline{\mu}_k)$ has μ-intercept $\overline{\eta}_k$ and slope $\overline{\nu}_k$ given by

$$\overline{\eta}_k = \mu_{\mathsf{mv}_k} - \frac{\nu_{\mathsf{as}_k}^2 \sigma_{\mathsf{mv}_k}^2}{\overline{\mu}_k - \mu_{\mathsf{mv}_k}}, \qquad \overline{\nu}_k = \frac{\nu_{\mathsf{as}_k}^2 \overline{\sigma}_k}{\overline{\mu}_k - \mu_{\mathsf{mv}_k}}.$$

If we set $\overline{\eta}_0 = -\infty$ then for every $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$ there is a unique j such that

$$\overline{\eta}_j \leq \mu_{\mathsf{si}} < \overline{\eta}_{j+1}.$$

For this value of j we have the tangancy parameters

$$\nu_{\mathsf{st}} = \nu_{\mathsf{as}_j} \sqrt{1 + \left(\frac{\mu_{\mathsf{mv}_j} - \mu_{\mathsf{si}}}{\nu_{\mathsf{as}_j} \sigma_{\mathsf{mv}_j}} \right)^2}, \quad \sigma_{\mathsf{st}} = \sigma_{\mathsf{mv}_j} \sqrt{1 + \left(\frac{\nu_{\mathsf{as}_j} \sigma_{\mathsf{mv}_j}}{\mu_{\mathsf{mv}_j} - \mu_{\mathsf{si}}} \right)^2}.$$

Therefore when $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$ the efficient long frontier is given by

$$\mu_{\mathsf{ef}}(\sigma) = \begin{cases} \mu_{\mathsf{si}} + \nu_{\mathsf{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\mathsf{st}}]\,, \\ \mu_{\mathsf{mv}_j} + \nu_{\mathsf{as}_j}\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_j}^2} & \text{for } \sigma \in [\sigma_{\mathsf{st}}, \overline{\sigma}_{j+1}]\,, \\ \mu_{\mathsf{mv}_k} + \nu_{\mathsf{as}_k}\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_k}^2} & \text{for } \sigma \in [\overline{\sigma}_k, \overline{\sigma}_{k+1}] \text{ and } k > j\,. \end{cases}$$

Because $\Gamma_{\mathsf{ef}}(\sigma) = \mu_{\mathsf{ef}}(\sigma) - \frac{1}{2}\sigma^2 - \chi\sigma$,

$$\Gamma_{\mathsf{ef}}'(\sigma) = \begin{cases} \nu_{\mathsf{st}} - \sigma - \chi & \text{for } \sigma \in [0, \sigma_{\mathsf{st}}]\,, \\ \dfrac{\nu_{\mathsf{as}_j}\,\sigma}{\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_j}^2}} - \sigma - \chi & \text{for } \sigma \in [\sigma_{\mathsf{st}}, \overline{\sigma}_{j+1}]\,, \\ \dfrac{\nu_{\mathsf{as}_k}\,\sigma}{\sqrt{\sigma^2 - \sigma_{\mathsf{mv}_k}^2}} - \sigma - \chi & \text{for } \sigma \in [\overline{\sigma}_k, \overline{\sigma}_{k+1}] \text{ and } k > j\,. \end{cases}$$

The last case in the above formulas can arise only when $\overline{\sigma}_{j+1} < \sigma_{\mathsf{mx}}$.

Therefore we find that

$$\sigma_{\mathsf{opt}} = \begin{cases} 0 & \text{if } \nu_{\mathsf{st}} \le \chi\,, \\ \nu_{\mathsf{st}} - \chi & \text{if } \nu_{\mathsf{st}} - \sigma_{\mathsf{st}} \le \chi < \nu_{\mathsf{st}}\,, \\ \sigma_{\mathsf{q}_j}(\chi) & \text{if } \overline{\nu}_{j+1} - \overline{\sigma}_{j+1} \le \chi < \nu_{\mathsf{st}} - \sigma_{\mathsf{st}}\,, \\ \sigma_{\mathsf{q}_k}(\chi) & \text{if } \overline{\nu}_{k+1} - \overline{\sigma}_{k+1} \le \chi < \overline{\nu}_k - \overline{\sigma}_k\,, \\ \sigma_{\mathsf{mx}} & \text{if } \chi < \nu_{\mathsf{mx}} - \sigma_{\mathsf{mx}}\,, \end{cases}$$

where $\sigma = \sigma_{\mathsf{q}_k}(\chi) \in [\overline{\sigma}_k, \overline{\sigma}_{k+1}]$ solves the quartic equation

$$\nu_{\mathsf{as}_k}^2\, \sigma^2 = \left(\sigma^2 - \sigma_{\mathsf{mv}_k}^2\right)(\sigma + \chi)^2\,.$$

The fourth case can arise only when $\overline{\sigma}_{j+1} < \sigma_{\mathsf{mx}}$.

Remark. The tasks of finding expressions for μ_{opt}, γ_{opt}, and $\mathbf{f}_{\mathsf{opt}}$ for the long portfolio model is left as an exercise.

Remark. The foregoing solutions illustrate two basic principles of investing.

When the market is bad it is often in the regime $\mu_{\mathsf{si}} \geq \eta_{\mathsf{mx}}$. In that case *the above solution gives an optimal long portfolio that is placed largely in the safe investment, but the part of the portfolio placed in risky assets is placed in the most agressive risky assets.* Such a position allows you to catch market upturns while putting little at risk when the market goes down.

When the market is good it is often in the regime $\mu_{\mathsf{si}} < \eta_{\mathsf{mx}}$. In that case *the above solution gives an optimal long portfolio that is placed largely in risky assets, but much of it is not placed in the most agressive risky assets.* Such a position protects you from market downturns while giving up little in returns when the market goes up.

Many investors will ignore these basic principles and become either overly conservative in a bear market or overly aggressive in a bull market.

Exercise. Consider the following groups of assets:

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Assume that μ_{si} is the US Treasury Bill rate at the end of the given year, and the μ_{cl} is three percentage points higher. Assume you are an investor who chooses $\chi = 0$. Design the optimal portfolios with risky assets drawn from group (a), from group (c), and from groups (a) and (c) combined. Do the same for group (b), group (d), and groups (b) and (d) combined. How well did these optimal portfolios actually do over the subsequent year?

Exercise. Repeat the above exercise for an investor who chooses $\chi = 1$. Compare these optimal portfolios with the corresponding ones from the previous exercise.