Modeling Portfolios that Contain Risky Assets

Optimization I: Model-Based Objective Functions

C. David Levermore
University of Maryland, College Park

Math 420: *Mathematical Modeling*
April 13, 2014 version

Optimization I: Model-Based Objective Functions

1. Law of Large Numbers
2. Growth Rate Mean
3. Central Limit Theorem
4. Growth Rate Exceeded with Probability
5. Risk Aversion

Optimization I: Model-Based Objective Functions

An IID model for the Markowitz portfolio with distribution $\mathbf{f}$ satisfies

$$\mathsf{Ex}\left(\log\left(\frac{\Pi(d)}{\Pi(0)}\right)\right) = \tfrac{d}{D}\,\gamma\,, \qquad \mathsf{Var}\left(\log\left(\frac{\Pi(d)}{\Pi(0)}\right)\right) = \tfrac{d}{D^2}\,\theta\,,$$

where γ and θ are estimated from a share price history by

$$\widehat{\gamma} = \mu_{\mathsf{rf}}\left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \mathbf{m}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,, \qquad \frac{\widehat{\theta}}{D} = \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,.$$

We see that $\widehat{\gamma}\,t$ is then the estimated expected growth of the IID model while $\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,t$ is its estimated variance at time $t = d/D$ years.

Our approach to portfolio management will be to select a distribution $\mathbf{f}$ that maximizes some objective function. Here we develop a family of such objective functions built from $\widehat{\gamma}$ and $\widehat{\theta}$ with the aid of two important tools from probability, the *Law of Large Numbers* and the *Central Limit Theorem.*

Law of Large Numbers. Let $\{X(d)\}_{d=1}^{\infty}$ be any sequence of IID random variables drawn from a probability density $p(X)$ with mean γ and variance $\theta > 0$. Let $\{Y(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y(d) = \frac{1}{d} \sum_{d'=1}^{d} X(d') \qquad \text{for every } d = 1, \cdots, \infty \, .$$

You can easily check that

$$\mathrm{Ex}(Y(d)) = \gamma \, , \qquad \mathrm{Var}(Y(d)) = \frac{\theta}{d} \, .$$

Given any $\delta > 0$ the *Law of Large Numbers* states that

$$\lim_{d \to \infty} \mathrm{Pr}\Big\{ |Y(d) - \gamma| \geq \delta \Big\} = 0 \, .$$

This limit is not uniform in δ. Its convergence rate can be estimated by the *Chebyshev inequality*, which yields the (not uniform in δ) upper bound

$$\mathrm{Pr}\Big\{ |Y(d) - \gamma| \geq \delta \Big\} \leq \frac{\mathrm{Var}(Y(d))}{\delta^2} = \frac{1}{\delta^2} \frac{\theta}{d} \, .$$

Remark. The Chebyshev inequality is easy to derive. Suppose that $p_d(Y)$ is the (unknown) probability density for $Y(d)$. Then

$$\begin{aligned}
\mathsf{Pr}\Big\{|Y(d)-\gamma| \geq \delta\Big\} &= \int_{\{Y\,:\,|Y-\gamma|\geq\delta\}} p_d(Y)\,\mathsf{d}Y \\
&\leq \int_{\{Y\,:\,|Y-\gamma|\geq\delta\}} \frac{|Y-\gamma|^2}{\delta^2}\, p_d(Y)\,\mathsf{d}Y \\
&\leq \frac{1}{\delta^2}\int |Y-\gamma|^2\, p_d(Y)\,\mathsf{d}Y = \frac{\mathsf{Var}(Y(d))}{\delta^2} = \frac{1}{\delta^2}\,\frac{\theta}{d}\,.
\end{aligned}$$

Remark. The unknown probability density $p_d(Y)$ can be expressed in terms of the unknown probability density $p(X)$ as

$$p_d(Y) = \int\cdots\int \delta\!\left(Y - \frac{1}{d}\sum_{d'=1}^{d} X_{d'}\right) p(X_1)\cdots p(X_d)\,\mathsf{d}X_1\cdots\mathsf{d}X_d\,,$$

where $\delta(\,\cdot\,)$ is the Dirac delta distribution introduced earlier.

Growth Rate Mean. Because the value of the associated portfolio is

$$\Pi(d) = \Pi(0)\,\exp\Big(Y(d)\tfrac{d}{D}\Big)\ ,$$

we see that $Y(d)$ is the growth rate of the portfolio at day d. The Law of Large Numbers implies that $Y(d)$ is likely to approach γ as $d \to \infty$. *This suggests that investors whose goal is to maximize the value of their portfolio over an extended period should maximize γ. More precisely, it suggests that such investors should select f to maximize the estimator $\widehat{\gamma}$.*

Remark. The suggestion to maximize $\widehat{\gamma}$ rests upon the assumption that the investor will hold the portfolio for an extended period. This is a suitable assumption for most young investors, but not for many old investors. The development of objective functions that are better suited for older investors requires more information about $Y(d)$ than the Law of Large Numbers provides. However, this additional information can be estimated with the aid of the *Central Limit Theorem.*

Central Limit Theorem. Let $\{X(d)\}_{d=1}^{\infty}$ be any sequence of IID random variables drawn from a probability density $p(X)$ with mean γ and variance $\theta > 0$. Let $\{Y(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y(d) = \frac{1}{d} \sum_{d'=1}^{d} X(d') \qquad \text{for every } d = 1, \cdots, \infty\,.$$

Recall that

$$\mathrm{Ex}(Y(d)) = \gamma\,, \qquad \mathrm{Var}(Y(d)) = \frac{\theta}{d}\,.$$

Now let $\{Z(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Z(d) = \frac{Y(d) - \gamma}{\sqrt{\theta/d}} \qquad \text{for every } d = 1, \cdots, \infty\,.$$

These random variables have been normalized so that

$$\mathrm{Ex}(Z(d)) = 0\,, \qquad \mathrm{Var}(Z(d)) = 1\,.$$

The *Central Limit Theorem* states that as $d \to \infty$ the limiting distribution of $Z(d)$ will be the mean-zero, variance-one normal distribution. Specifically, for every $\zeta \in \mathbb{R}$ it implies that

$$\lim_{d\to\infty} \Pr\Big\{Z(d) \geq -\zeta\Big\} = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\, \mathrm{d}Z\,.$$

This can be expressed in terms of $Y(d)$ as

$$\lim_{d\to\infty} \Pr\Big\{Y(d) \geq \gamma - \zeta\sqrt{\theta/d}\Big\} = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\, \mathrm{d}Z\,.$$

Remark. The power of the Central Limit Theorem is that it assumes so little about the underlying probability density $p(X)$. Specifically, it assumes that

$$\int_{-\infty}^{\infty} X^2 p(X)\, \mathrm{d}X < \infty\,,$$

and that

$$0 < \theta = \int_{-\infty}^{\infty} (X-\gamma)^2 p(X)\, \mathrm{d}X\,, \quad \text{where} \quad \gamma = \int_{-\infty}^{\infty} X\, p(X)\, \mathrm{d}X\,.$$

Remark. The Central Limit Theorem does not estimate how fast this limit is approached. Any such estimate would require additional assumptions about the underlying probability density $p(X)$. Roughly speaking, the rate of convergence will be slower when $p(X)$ is further from a normal density.

Remark. In an IID model of a portfolio $Y(d)$ is the growth rate of the portfolio when it is held for d days. The Central Limit Theorem shows that as $d \to \infty$ the values of $Y(d)$ become strongly peak around γ. This behavior seems to be consistent with the idea that a reasonable approach towards portfolio management is to select $\mathbf{f}$ to maximize the estimator $\widehat{\gamma}$. However, by taking $\zeta = 0$ we see that the Central Limit Theorem implies

$$\lim_{d\to\infty} \mathrm{Pr}\Big\{Y(d) \geq \gamma\Big\} = \tfrac{1}{2}\,.$$

This shows that in the long run the growth rate of a portfolio will exceed γ with a probability of only $\frac{1}{2}$. A conservative investor might want the portfolio to exceed the optimized growth rate with a higher probability.

Growth Rate Exceeded with Probability. Let $\Gamma(\lambda, T)$ be the growth rate exceeded by a portfolio with probability λ at time T in years. *Here we will use the Central Limit Theorem to construct an estimator $\widehat{\Gamma}(\lambda, T)$ of this quantity. We do this by assuming $T = d/D$ is large enough that we can use the approximation*

$$\mathrm{Pr}\left\{ Y(d) \geq \gamma - \zeta\sqrt{\theta/d} \right\} \approx \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}Z^2} \, \mathrm{d}Z \, .$$

Given any probability $\lambda \in (0, 1)$, we set

$$\lambda = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}Z^2} \, \mathrm{d}Z = \int_{-\infty}^{\zeta} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}Z^2} \, \mathrm{d}Z \equiv \mathrm{N}(\zeta) \, .$$

Our approximation can then be expressed as

$$\mathrm{Pr}\left\{ Y(d) \geq \gamma - \frac{\zeta}{\sqrt{T}} \, \sigma \right\} \approx \lambda \, ,$$

where $\sigma = \sqrt{\theta/D}$ and $\zeta = \mathrm{N}^{-1}(\lambda)$.

Finally, we replace γ and σ in the above approximation by the estimators

$$\widehat{\gamma} = \mu_{\mathsf{rf}} \left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \mathbf{m}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,, \qquad \widehat{\sigma} = \sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,.$$

This yields the estimator

$$\widehat{\Gamma}(\lambda, T) = \widehat{\gamma} - \frac{\zeta}{\sqrt{T}}\,\widehat{\sigma} = \widehat{\mu} - \tfrac{1}{2}\widehat{\sigma}^2 - \frac{\zeta}{\sqrt{T}}\,\widehat{\sigma}\,,$$

where $\widehat{\mu} = \mu_{\mathsf{rf}}(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}) + \mathbf{m}^{\mathsf{T}}\mathbf{f}$ and $\zeta = \mathsf{N}^{-1}(\lambda)$.

Remark. *The only new assumption we have made in order to construct this estimator is that T is large enough for the Central Limit Theorem to yield a good approximation of the distribution of growth rates.* Investors often choose T to be the interval at which the portfolio will be rebalanced, regardless of whether T is large enough for the approximation to be valid. If an investor plans to rebalance once a year then $T = 1$, twice a year then $T = \frac{1}{2}$, and four times a year then $T = \frac{1}{4}$. The smaller T, the less likely it is that the Central Limit Theorem approximation is valid.

Risk Aversion. *The idea now will be to select the admissible Markowitz portfilio that maximizes $\widehat{\Gamma}(\lambda, T)$ given a choice of λ and T by the investor. In other words, the objective will be to maximize the growth rate that will be exceeded by the portfolio with probability λ when it is held for T years.* Because $1 - \lambda$ is the fraction of times the investor is willing to experience a downside tail event, the choice of λ measures the *risk aversion* of the investor. More risk averse investors will select a higher λ.

Remark. The risk aversion of an investor generally increases with age. Retirees whose portfolio provides them with an income that covers much of their living expenses will generally be extremely risk averse. Investors within ten years of retirement will be fairly risk averse because they have less time for their nest-egg to recover from any economic downturn. In constrast, young investors can be less risk averse because they have more time to experience economic upturns and because they are typically far from their peak earning capacity.

An investor can simply select ζ such that $\lambda = \mathrm{N}(\zeta)$ is a probability that reflects their risk aversion. For example, based on the tabulations

$$
\begin{aligned}
&\mathrm{N}(0) = .5000\,, \quad \mathrm{N}\big(\tfrac{1}{4}\big) \approx .5987\,, \quad \mathrm{N}\big(\tfrac{1}{2}\big) \approx .6915\,, \quad \mathrm{N}\big(\tfrac{3}{4}\big) \approx .7734\,, \\
&\mathrm{N}(1) \approx .8413\,, \quad \mathrm{N}\big(\tfrac{5}{4}\big) \approx .8944\,, \quad \mathrm{N}\big(\tfrac{3}{2}\big) \approx .9332\,, \quad \mathrm{N}\big(\tfrac{7}{4}\big) \approx .9505\,,
\end{aligned}
$$

an investor who is willing to experience a downside tail event roughly

$$
\begin{aligned}
\text{once every two years might select } \zeta &= 0\,, \\
\text{twice every five years might select } \zeta &= \tfrac{1}{4}\,, \\
\text{thrice every ten years might select } \zeta &= \tfrac{1}{2}\,, \\
\text{twice every nine years might select } \zeta &= \tfrac{3}{4}\,, \\
\text{once every six years might select } \zeta &= 1\,, \\
\text{once every ten years might select } \zeta &= \tfrac{5}{4}\,, \\
\text{once every fifteen years might select } \zeta &= \tfrac{3}{2}\,, \\
\text{once every twenty years might select } \zeta &= \tfrac{7}{4}\,.
\end{aligned}
$$

Remark. *The Central Limit Theorem approximation generally degrades badly as ζ increases because $p(X)$ typically decays much more slowly than a normal density as $X \to -\infty$.* Therefore it is a bad idea to pick $\zeta > 2$ based on this approximation. Fortunately, $\zeta = \frac{7}{4}$ already corresponds to a fairly conservative investor.

Remark. *You should pick a larger value of ζ whenever your analysis of the historical data gives you less confidence either in the calibration of $\mathbf{m}$ and $\mathbf{V}$ or in the validity of an IID model.*

Remark. Our approach is similar to something in financal management called *value at risk*. The finance problem is much harder because the time horizon T considered there is much shorter, typically on the order of days. In that setting the Central Limit Theorem approximation is certainly invalid.