Modeling Portfolios that Contain Risky Assets

Stochastic Models III: Growth Rates for Portfolios

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1. Moment and Cumulant Generating Functions
2. Estimators for Growth Rate Mean and Variance
3. Uncertainty in the Interpolation Estimators

Stochastic Models III: Growth Rates for Portfolios

The idea now is to treat the Markowitz portfolio associated with f as a single risky asset that can be modeled by the IID process associated with the growth rate probability density $p_{\mathrm{f}}(X)$ given by

$$p_{\mathrm{f}}(X) = q_{\mathrm{f}}\left(D\left(e^{\frac{1}{D}X} - 1\right)\right) e^{\frac{1}{D}X} .$$

The mean γ and variance θ of X are given by

$$\gamma = \int X \, p_{\mathrm{f}}(X) \, \mathrm{d}X , \qquad \theta = \int (X - \gamma)^2 p_{\mathrm{f}}(X) \, \mathrm{d}X .$$

We know from our study of one risky asset that γ is a good proxy for reward, while $\sqrt{\frac{1}{D}\theta}$ is a good proxy for risk. Therefore we would like to estimate γ and θ in terms of the estimators $\widehat{\mu}$ and $\widehat{\xi}$ that we studied last time.

Moment and Cumulant Generating Functions. Estimators for γ and θ will be constructed from the positive function

$$M(\tau) = \mathsf{Ex}\Big(e^{\tau X}\Big) = \int e^{\tau X}\, p_{\mathbf{f}}(X)\, \mathsf{d}X\,.$$

We will assume $M(\tau)$ is defined for every τ in an open interval $(\tau_{\mathsf{mn}}, \tau_{\mathsf{mx}})$ that contains the interval $[0, \frac{2}{D}]$. It can then be shown that $M(\tau)$ is infinitely differentiable over $(\tau_{\mathsf{mn}}, \tau_{\mathsf{mx}})$ with

$$M^{(m)}(\tau) = \mathsf{Ex}\Big(X^m\, e^{\tau X}\Big) = \int X^m\, e^{\tau X}\, p_{\mathbf{f}}(X)\, \mathsf{d}X\,.$$

We call $M(\tau)$ the *moment generating function* for X because, by setting $\tau = 0$ in the above expression, we see that the *moments* $\{\mathsf{Ex}(X^m)\}_{m=1}^{\infty}$ are generated from $M(\tau)$ by the formula

$$\mathsf{Ex}\Big(X^m\Big) = \int X^m p_{\mathbf{f}}(X)\, \mathsf{d}X = M^{(m)}(0)\,.$$

A related inifinitely differentiable function over $(\tau_{\mathsf{mn}}, \tau_{\mathsf{mx}})$ is

$$K(\tau) = \log(M(\tau)) = \log\Big(\mathsf{Ex}\big(e^{\tau X}\big)\Big) \,.$$

We call $K(\tau)$ the *cumulant generating function* because the *cumulants* $\{\kappa_m\}_{m=1}^{\infty}$ of X are generated by the formula $\kappa_m = K^{(m)}(0)$. Because

$$K'(\tau) = \frac{\mathsf{Ex}\big(X\, e^{\tau X}\big)}{\mathsf{Ex}\big(e^{\tau X}\big)}\,,$$

$$K''(\tau) = \frac{\mathsf{Ex}\Big((X - K'(\tau))^2 e^{\tau X}\Big)}{\mathsf{Ex}\big(e^{\tau X}\big)}\,,$$

$$K'''(\tau) = \frac{\mathsf{Ex}\Big((X - K'(\tau))^3 e^{\tau X}\Big)}{\mathsf{Ex}\big(e^{\tau X}\big)}\,,$$

$$K''''(\tau) = \frac{\mathsf{Ex}\Big((X - K'(\tau))^4 e^{\tau X}\Big)}{\mathsf{Ex}\big(e^{\tau X}\big)} - 3K''(\tau)^2\,,$$

we see that the first four cumulants of X are

$$
\begin{aligned}
\kappa_1 = &\quad K'(0) = \mathrm{Ex}(X) = \gamma\,, \\
\kappa_2 = &\quad K''(0) = \mathrm{Ex}\Big((X-\gamma)^2\Big) = \theta\,, \\
\kappa_3 = &\quad K'''(0) = \mathrm{Ex}\Big((X-\gamma)^3\Big)\,, \\
\kappa_4 = &\quad K''''(0) = \mathrm{Ex}\Big((X-\gamma)^4\Big) - 3\theta^2\,.
\end{aligned}
$$

We call these respectively the mean, variance, skewness, and kurtosis. The skewness measures asymmetry in the tails of the distribution. It is positive or negative depending on whether the fatter tail is to the right or left respectively. The kurtosis measures a relationship between the tails and the center of the distribution. It is greater for distributions with greater weight in the tails than in the center.

Remark. It should be evident from the formulas on the previous slide that $K'(\tau)$, $K''(\tau)$, $K'''(\tau)$, and $K''''(\tau)$ are the mean, variance, skewness, and kurtosis for the probability density $e^{\tau X} p_{\mathrm{f}}(X)/\mathrm{Ex}(e^{\tau X})$.

Remark. If X is normally distributed with mean γ and variance θ then

$$p_{\mathrm{f}}(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) .$$

A direct calculation then shows that

$$\begin{aligned}
\mathrm{Ex}\left(e^{\tau X}\right) &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma)^2}{2\theta} + \tau X\right) \, \mathrm{d}X \\
&= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma-\tau\theta)^2}{2\theta} + \tau\gamma + \tfrac{1}{2}\tau^2\theta\right) \, \mathrm{d}X \\
&= \exp\left(\tau\gamma + \tfrac{1}{2}\tau^2\theta\right) ,
\end{aligned}$$

Therefore $K(\tau) = \log(\mathrm{Ex}(e^{\tau X})) = \tau\gamma + \frac{1}{2}\tau^2\theta$. *This shows that when X is normally distributed the skewness, kurtosis, and all other higher-order cumulants vanish. Conversely, if all these higher-order cumulants vanish then X is normally distributed.*

Remark. The cumulent generating function $K(\tau)$ is *strictly convex* over the interval $(\tau_{\text{mn}}, \tau_{\text{mx}})$ because $K''(\tau) > 0$.

Remark. We can also see that $K(\tau)$ is convex over $(\tau_{\text{mn}}, \tau_{\text{mx}})$ as follows. Let τ_0, $\tau_1 \in (\tau_{\text{mn}}, \tau_{\text{mx}})$. By applying the *Hölder inequality* with $p = \frac{1}{1-s}$ and $p^* = \frac{1}{s}$, we see that for every $s \in (0, 1)$ we have

$$
\begin{aligned}
M\Big((1-s)\tau_0 + s\tau_1)\Big) &= \int e^{(1-s)\tau_0 X}\, e^{s\tau_1 X}\, p_{\mathbf{f}}(X)\, \mathrm{d}X \\
&\leq \left(\int e^{\tau_0 X} p_{\mathbf{f}}(X)\, \mathrm{d}X\right)^{1-s} \left(\int e^{\tau_1 X} p_{\mathbf{f}}(X)\, \mathrm{d}X\right)^{s} \\
&= M(\tau_0)^{1-s} M(\tau_1)^{s} \,.
\end{aligned}
$$

By taking the logarithm of this inequality we obtain

$$
K((1-s)\tau_0 + s\tau_1) \leq (1-s)K(\tau_0) + sK(\tau_1) \quad \text{for every } s \in (0, 1)\,.
$$

Therefore $K(\tau)$ is a convex function over $(\tau_{\text{mn}}, \tau_{\text{mx}})$.

Estimators for Growth Rate Mean and Variance. We will now construct estimators for γ and θ by using the moment generating function

$$M(\tau) = \mathsf{Ex}\Big(e^{\tau X}\Big) \,.$$

Because $R = D(e^{\frac{1}{D}X} - 1)$ and $\mathsf{Ex}(e^{\frac{1}{D}X}) = M(\frac{1}{D})$, we have

$$\mu = \mathsf{Ex}(R) = D\left(M(\tfrac{1}{D}) - 1\right) \,.$$

Because $R - \mu = D\Big(e^{\frac{1}{D}X} - M(\frac{1}{D})\Big)$ and $\mathsf{Ex}(e^{\frac{2}{D}X}) = M(\frac{2}{D})$, we have

$$\xi = \mathsf{Ex}\Big((R-\mu)^2\Big) = D^2\left(M(\tfrac{2}{D}) - M(\tfrac{1}{D})^2\right) \,.$$

These equations can be solved for $M(\frac{1}{D})$ and $M(\frac{2}{D})$ as

$$M(\tfrac{1}{D}) = 1 + \frac{\mu}{D}\,, \qquad M(\tfrac{2}{D}) = \left(1 + \frac{\mu}{D}\right)^2 + \frac{\xi}{D^2}\,.$$

Therefore knowing μ and ξ is equivalent to knowing $M(\frac{1}{D})$ and $M(\frac{2}{D})$.

Because $\text{Ex}(X) = M'(0)$ and $\text{Ex}(X^2) = M''(0)$, we see that

$$\gamma = \text{Ex}(X) = M'(0)\,,$$
$$\theta = \text{Ex}\Big((X-\gamma)^2\Big) = \text{Ex}\Big(X^2\Big) - \gamma^2 = M''(0) - M'(0)^2\,.$$

Because $M(0) = 1$, we construct an estimator of $M(\tau)$ by interpolating the values $M(0)$, $M(\frac{1}{D})$, and $M(\frac{2}{D})$ with a quadratic polynomial as

$$\begin{aligned}\widehat{M}(\tau) &= 1 + \tau D\Big(M(\tfrac{1}{D}) - 1\Big) + \tau\Big(\tau - \tfrac{1}{D}\Big)\tfrac{D^2}{2}\Big(M(\tfrac{2}{D}) - 2M(\tfrac{1}{D}) + 1\Big) \\ &= 1 + \tau\mu + \tfrac{1}{2}\tau\left(\tau - \tfrac{1}{D}\right)\left(\mu^2 + \xi\right)\,.\end{aligned}$$

By direct calculation we see that

$$\widehat{M}'(0) = \mu - \tfrac{1}{2D}(\mu^2 + \xi)\,, \qquad \widehat{M}''(0) = \mu^2 + \xi\,.$$

We then construct estimators $\widehat{\gamma}$ and $\widehat{\theta}$ as functions of μ and ξ by

$$\begin{aligned}\widehat{\gamma} &= \widehat{M}'(0) \\ &= \mu - \tfrac{1}{2D}(\mu^2 + \xi)\,,\end{aligned} \qquad \begin{aligned}\widehat{\theta} &= \widehat{M}''(0) - \widehat{M}'(0)^2 \\ &= \mu^2 + \xi - \left(\mu - \tfrac{1}{2D}(\mu^2 + \xi)\right)^2.\end{aligned}$$

By replacing the μ and ξ that appear in the foregoing estimators for $\widehat{\gamma}$ and $\widehat{\theta}$ with the estimators $\widehat{\mu} = \mu_{\mathsf{rf}}(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}) + \mathbf{m}^{\mathsf{T}}\mathbf{f}$ and $\widehat{\xi} = D\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$, we obtain the new estimators

$$
\begin{aligned}
\widehat{\gamma} &= \widehat{\mu} - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \tfrac{1}{2D}\widehat{\mu}^2\,,\\
\widehat{\theta} &= \widehat{\mu}^2 + D\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \left(\widehat{\mu} - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \tfrac{1}{2D}\widehat{\mu}^2\right)^2\,.
\end{aligned}
$$

Finally, if we assume D is large in the sense that

$$
\left|\frac{\widehat{\mu}}{D}\right| \ll 1\,, \qquad \left|\frac{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}{D}\right| \ll 1\,,
$$

then, *by keeping the leading order of each term, we arrive at the estimators*

$$
\widehat{\gamma} = \mu_{\mathsf{rf}}\left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \mathbf{m}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,, \qquad \frac{\widehat{\theta}}{D} = \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,.
$$

The error of this last approximation can be examined by simply comparing the result of these estimators with that of those at the top of this page.

Remark. The estimators $\widehat{\gamma}$ and $\widehat{\theta}$ given above have at least three potential sources of error:

- the "large D" approximation made at the bottom of the previous page,
- the estimators $\widehat{M}'(0)$ and $\widehat{M}''(0)$ as functions of μ and ξ,
- the estimators $\widehat{\mu}$ and $\widehat{\xi}$ used to approximate μ and ξ.

These approximations all assume that the return rate distribution for each Markowitz portfolio is described by a density $q_{\mathbf{f}}(\mathbf{R})$ that is narrow enough for some moment beyond the second to exist. All of these approximations should be examined carefully, especially when markets are highly volatile. The first was examined at the bottom of the last slide. The second will be examined in the next section. The third was examined last time.

We now give an alternative derivation of these estimators that uses the cumulent generating function $K(\tau) = \log(M(\tau))$ and is based on the fact that $\gamma = K'(0)$ and $\theta = K''(0)$. It begins by observing that

$$K(\tfrac{1}{D}) = \log\Big(M(\tfrac{1}{D})\Big) = \log\left(1 + \frac{\mu}{D}\right) ,$$
$$K(\tfrac{2}{D}) = \log\Big(M(\tfrac{2}{D})\Big) = \log\left(\left(1 + \frac{\mu}{D}\right)^2 + \frac{\xi}{D^2}\right) .$$

Therefore knowing μ and ξ is equivalent to knowing $K(\frac{1}{D})$ and $K(\frac{2}{D})$.

Because $K(0) = 0$, we construct an estimator of $K(\tau)$ by interpolating the values $K(0)$, $K(\frac{1}{D})$, and $K(\frac{2}{D})$ with a quadratic polynomial as

$$\begin{aligned}\widehat{K}(\tau) &= \tau D K(\tfrac{1}{D}) + \tau\Big(\tau - \tfrac{1}{D}\Big)\tfrac{D^2}{2}\Big(K(\tfrac{2}{D}) - 2K(\tfrac{1}{D})\Big) \\ &= \tau D \log\left(1 + \frac{\mu}{D}\right) + \tau\Big(\tau - \tfrac{1}{D}\Big)\tfrac{D^2}{2}\log\left(1 + \frac{\xi}{(D+\mu)^2}\right) .\end{aligned}$$

This yields the estimators

$$\widehat{\gamma} = \widehat{K}'(0) = D \log\left(1 + \frac{\mu}{D}\right) - \tfrac{1}{2} D \log\left(1 + \frac{\xi}{(D+\mu)^2}\right) \,,$$
$$\widehat{\theta} = \widehat{K}''(0) = D^2 \log\left(1 + \frac{\xi}{(D+\mu)^2}\right) \,.$$

By replacing the μ and ξ that appear in the above estimators for $\widehat{\gamma}$ and $\widehat{\theta}$ with the estimators $\widehat{\mu} = \mu_{\mathsf{rf}}(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}) + \mathbf{m}^{\mathsf{T}}\mathbf{f}$ and $\widehat{\xi} = D\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$, we obtain the new estimators

$$\widehat{\gamma} = D \log\left(1 + \frac{\widehat{\mu}}{D}\right) - \tfrac{1}{2} D \log\left(1 + \frac{D\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}{(D+\widehat{\mu})^2}\right) \,,$$
$$\widehat{\theta} = D^2 \log\left(1 + \frac{D\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}{(D+\widehat{\mu})^2}\right) \,.$$

Finally, if we assume D is large in the sense that

$$\left|\frac{\widehat{\mu}}{D}\right| \ll 1\,, \qquad \left|\frac{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}{D}\right| \ll 1\,,$$

then, *by keeping the leading order of each term, we arrive at the estimators*

$$\widehat{\gamma} = \mu_{\mathsf{rf}}\left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \mathbf{m}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,, \qquad \widehat{\theta} = D\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,.$$

Remark. *These are the same estimators that we obtained from our first derivation. The fact that both derivations lead to the same result gives us greater confidence in the validity of these estimators in the large D regime.*

Remark. If the Markowitz portfolio specified by $\mathbf{f}$ has growth rates X that are normally distributed with mean γ and variance θ then we have seen that $K(\tau) = \tau\gamma + \frac{1}{2}\tau^2\theta$. In this case we have $\widehat{K}(\tau) = K(\tau)$, so the estimators $\widehat{\gamma} = \widehat{K}'(0)$ and $\widehat{\theta} = \widehat{K}''(0)$ are exact.

Uncertainty in the Interpolation Estimators. Here we examine the errors of the interpolation-based estimators given by

$$\widehat{M}'(0) = D\Big(2\Big(M(\tfrac{1}{D}) - 1\Big) - \tfrac{1}{2}\Big(M(\tfrac{2}{D}) - 1\Big)\Big)\,,$$
$$\widehat{M}''(0) = D^2\Big(M(\tfrac{2}{D}) - 2M(\tfrac{1}{D}) + 1\Big)\,.$$

Let $M(\tau)$ be any thrice continuously differentiable function over $[0, \frac{2}{D}]$ that satisfies $M(0) = 1$. The Cauchy form of the Taylor remainder then yields

$$M(\tfrac{1}{D}) = 1 + \tfrac{1}{D}M'(0) + \tfrac{1}{2D^2}M''(0) + \tfrac{1}{2}\int_0^{\frac{1}{D}} \Big(\tfrac{1}{D} - s\Big)^2 M'''(s)\,\mathrm{d}s\,,$$
$$M(\tfrac{2}{D}) = 1 + \tfrac{2}{D}M'(0) + \tfrac{2}{D^2}M''(0) + \tfrac{1}{2}\int_0^{\frac{2}{D}} \Big(\tfrac{2}{D} - s\Big)^2 M'''(s)\,\mathrm{d}s\,.$$

By placing these into the above formulas for $\widehat{M}'(0)$ and $\widehat{M}''(0)$ we obtain

$$\widehat{M}'(0) = M'(0) + E_1\,, \qquad \widehat{M}''(0) = M''(0) + E_2\,,$$

where the errors E_1 and E_2 are given by

$$\begin{aligned}
E_1 &= D\bigg[\int_0^{\frac{1}{D}}\left(\tfrac{1}{D}-s\right)^2 M'''(s)\,\mathrm{d}s - \tfrac{1}{4}\int_0^{\frac{2}{D}}\left(\tfrac{2}{D}-s\right)^2 M'''(s)\,\mathrm{d}s\bigg] \\
&= -D\bigg[\int_0^{\frac{1}{D}}\left(\tfrac{1}{D}s-\tfrac{3}{4}s^2\right) M'''(s)\,\mathrm{d}s + \tfrac{1}{4}\int_{\frac{1}{D}}^{\frac{2}{D}}\left(\tfrac{2}{D}-s\right)^2 M'''(s)\,\mathrm{d}s\bigg], \\
E_2 &= D^2\bigg[\tfrac{1}{2}\int_0^{\frac{2}{D}}\left(\tfrac{2}{D}-s\right)^2 M'''(s)\,\mathrm{d}s - \int_0^{\frac{1}{D}}\left(\tfrac{1}{D}-s\right)^2 M'''(s)\,\mathrm{d}s\bigg] \\
&= D^2\bigg[\tfrac{1}{2}\int_{\frac{1}{D}}^{\frac{2}{D}}\left(\tfrac{2}{D}-s\right)^2 M'''(s)\,\mathrm{d}s + \int_0^{\frac{1}{D}}\left(\tfrac{1}{D^2}-\tfrac{1}{2}s^2\right) M'''(s)\,\mathrm{d}s\bigg].
\end{aligned}$$

Here the integrals seen in the second expression for each error are written so that the factor multiplying $M'''(s)$ inside each integral is nonnegative. This shows that if $M'''(s) \geq 0$ over $[0, \frac{2}{D}]$ then $E_1 < 0$ and $E_2 > 0$, while if $M'''(s) \leq 0$ over $[0, \frac{2}{D}]$ then $E_1 > 0$ and $E_2 < 0$.

The errors E_1 and E_2 may be bounded in terms of

$$\|M'''\|_\infty = \max\left\{|M'''(\tau)| \,:\, \tau \in [0, \tfrac{2}{D}]\right\}.$$

Specifically, because

$$\int_0^{\frac{1}{D}} \left(\tfrac{1}{D}s - \tfrac{3}{4}s^2\right) \mathrm{d}s = \tfrac{1}{4D^3}, \qquad \int_{\frac{1}{D}}^{\frac{2}{D}} \left(\tfrac{2}{D} - s\right)^2 \mathrm{d}s = \tfrac{1}{3D^3},$$

$$\int_0^{\frac{1}{D}} \left(\tfrac{1}{D^2} - \tfrac{1}{2}s^2\right) \mathrm{d}s = \tfrac{5}{6D^3},$$

we obtain the bounds

$$|E_1| \le \tfrac{1}{3D^2}\|M'''\|_\infty, \qquad |E_2| \le \tfrac{1}{D}\|M'''\|_\infty.$$

This shows that the estimators $\widehat{M}'(0)$ and $\widehat{M}''(0)$ have errors that are $O(\frac{1}{D^2})$ and $O(\frac{1}{D})$ respectively.

If we want to use these error bounds then we must find either a bound of or an approximation to $\|M'''\|_\infty$. From the definition of $M(\tau)$ we see that

$$M'''(\tau) = \mathsf{Ex}\Big(X^3 e^{\tau X}\Big) = \int X^3 e^{\tau X} p_{\mathbf{f}}(X)\,\mathrm{d}X\,.$$

Because

$$M''''(\tau) = \mathsf{Ex}\Big(X^4 e^{\tau X}\Big) = \int X^4 e^{\tau X} p_{\mathbf{f}}(X)\,\mathrm{d}X > 0\,,$$

we see that $M'''(\tau)$ is a strictly increasing function of τ. Therefore

$$\|M'''\|_\infty = \max\Big\{ -M'''(0)\,,\, M'''(\tfrac{2}{D})\Big\}\,,$$

where the quantities $M'''(0)$ and $M'''(\frac{2}{D})$ can be expressed in terms of the return rate density as

$$\begin{aligned}
M'''(0) &= \int_{-D}^{\infty} \Big(D\log\Big(1+\tfrac{1}{D}R\Big)\Big)^3 q_{\mathbf{f}}(R)\,\mathrm{d}R\,,\\
M'''(\tfrac{2}{D}) &= \int_{-D}^{\infty} \Big(D\log\Big(1+\tfrac{1}{D}R\Big)\Big)^3 \Big(1+\tfrac{1}{D}R\Big)^2 q_{\mathbf{f}}(R)\,\mathrm{d}R\,.
\end{aligned}$$

These quantities can be approximated by the sample means

$$\widetilde{M'''}(0) = \sum_{d=1}^{D_h} w(d) \left(D \log\!\left(1 + \tfrac{1}{D} r(d)\right)\right)^3 ,$$

$$\widetilde{M'''}(\tfrac{2}{D}) = \sum_{d=1}^{D_h} w(d) \left(D \log\!\left(1 + \tfrac{1}{D} r(d)\right)\right)^3 \left(1 + \tfrac{1}{D} r(d)\right)^2 ,$$

where $\{r(d)\}_{d=1}^{D_h}$ is the portfolio return rate history given by

$$r(d) = (1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{f}^{\mathsf{T}}\mathbf{r}(d) \,.$$

By arguing as we did for $M'''(\tau)$, we can show that $\widetilde{M'''}(0) < \widetilde{M'''}(\frac{2}{D})$. Therefore we can approximate $\|M'''\|_\infty$ by

$$\|M'''\|_\infty \approx \max\Big\{ -\widetilde{M'''}(0) \,,\, \widetilde{M'''}(\tfrac{2}{D})\Big\} \,.$$

This approximation can be used to quantify the uncertainty associated with the interpolation-based estimators $\widehat{M}'(0)$ and $\widehat{M}''(0)$.

Exercise. When the final forms of the estimators $\widehat{\gamma}$ and $\widehat{\theta}$ are applied to a single risky asset, they reduce to

$$\widehat{\gamma} = \widehat{\mu} - \tfrac{1}{2D}\widehat{\xi}\,, \qquad \widehat{\theta} = \widehat{\xi}\,.$$

Use these to estimate γ and θ for each of the following assets given the share price history $\{s(d)\}_{d=0}^{D}$. How do these $\widehat{\gamma}$ and $\widehat{\theta}$ compare with the unbiased estimators for γ and θ that you obtained in the previous problem?

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Exercise. Compute $\widehat{\gamma}$ and $\widehat{\theta}$ based on daily data for the Markowitz portfolio with value equally distributed among the assets in each of the groups given in the previous exercise.