Modeling Portfolios that Contain Risky Assets

Stochastic Models I: One Risky Asset

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Risk and Reward I: Introduction
II: Markowitz Portfolios
III: Basic Markowitz Portfolio Theory
Portfolio Models I: Portfolios with Risk-Free Assets
II: Long Portfolios
III: Long Portfolios with a Safe Investment
Stochastic Models I: One Risky Asset
II: Portfolios with Risky Assets
III: Growth Rates for Portfolios
Optimization I: Model-Based Objective Functions
II: Model-Based Target Portfolios
III: Conclusion

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1. IID Models for an Asset
2. Return Rate Probability Densities
3. Growth Rate Probability Densities
4. Normal Growth Rate Model

Stochastic Models I: One Risky Asset

Investors have long followed the old adage "don't put all your eggs in one basket" by holding diversified portfolios. However, before MPT the value of diversification had not been quantified. Key aspects of MPT are:

1. it uses the return rate mean as a proxy for reward;
2. it uses volatility as a proxy for risk;
3. it analyzes Markowitz portfolios;
4. it shows diversification reduces volatility through covariances;
5. it identifies the efficient frontier as the place to be.

The orignial form of MPT did not give guidance to investors about where to be on the efficient frontier. We will now begin to build stochasitc models that can be used in conjunction with the original MPT to address this question. *By doing so, we will see that maximizing the return rate mean is not the best strategy for maximizing your reward.*

IID Models for an Asset. We begin by building models of one risky asset with a share price history $\{s(d)\}_{d=0}^{D_h}$. Let $\{r(d)\}_{d=1}^{D_h}$ be the associated return rate history. Because each $s(d)$ is positive, each $r(d)$ lies in the interval $(-D, \infty)$. An *independent, identically-distributed (IID)* model for this history simply independently draws D_h random numbers $\{R(d)\}_{d=1}^{D_h}$ from $(-D, \infty)$ in accord with a fixed probability density $q(R)$ over $(-D, \infty)$. This means that $q(R)$ is a nonnegative integrable function such that

$$\int_{-D}^{\infty} q(R)\,\mathrm{d}R = 1\,,$$

and that the probability that each $R(d)$ takes a value inside any interval $[R_1, R_2] \subset (-D, \infty)$ is given by

$$\mathrm{Pr}\Big\{R(d) \in [R_1, R_2]\Big\} = \int_{R_1}^{R_2} q(R)\,\mathrm{d}R\,.$$

Here capitol letters $R(d)$ denote random numbers drawn from $(-D, \infty)$ in accord with the probability density $q(R)$ rather than real return rate data.

Because the random numbers $\{R(d)\}_{d=1}^{D_h}$ are drawn from $(-D, \infty)$ in accord with the probability density $q(R)$ *independent* of each other, there is no correlation of $R(d)$ with $R(d')$ when $d \neq d'$. In particular, if we plot the points $\{(R(d), R(d+c))\}_{d=1}^{D_h - c}$ in the rr'-plane for any $c > 0$ they will be distributed in accord with the probability density $q(r)q(r')$. *Therefore if the return rate history $\{r(d)\}_{d=1}^{D_h}$ is mimiced by such a model then the points $\{(r(d), r(d+c))\}_{d=1}^{D_h - c}$ plotted in the rr'-plane should appear to be distributed in a way consistant with the probability density $q(r)q(r')$.* Such plots are called *scatter plots.*

In general a scatter plot will not show independence when c is small. This is because the behavior of an asset on any given trading day generally correlates with its behavior on the previous trading day. However, if a scatter plot shows independence for some c that is small compared to D_h the an IID model might still be good. Such a time c is called the *correlation time.*

Because the random numbers $\{R(d)\}_{d=1}^{D_h}$ are each drawn from $(-D, \infty)$ in accord with the *same* probability density $q(R)$, if we plot the points $\{(d, R(d))\}_{d=1}^{D_h}$ in the dr-plane they will usually be distributed in a way that looks uniform in d. *Therefore if the return rate history $\{r(d)\}_{d=1}^{D_h}$ is mimiced by such a model then the points $\{(d, r(d))\}_{d=1}^{D_h}$ plotted in the dr-plane should appear to be distributed in a way that is unifrom in d.*

Exercise. Plot $\{(r(d), r(d+1))\}_{d=1}^{D_h-1}$ and $\{(d, r(d))\}_{d=1}^{D_h}$ for each of the following assets and explain which might be good candidates to be mimiced by an IID model.

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2013;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2008;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2013;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2008.

Remark. We have adopted IID models because they are simple. It is not hard to develop more complicated stochastic models. For example, we could use a different probability density for each day of the week rather than treating all trading days the same way. Because there are usually five trading days per week, Monday through Friday, such a model would require calibrating five times as many means and covariances with one fifth as much data. There would then be greater uncertainty associated with the calibration. Moreover, we then have to figure out how to treat weeks that have less than five trading days due to holidays. Perhaps just the first and last trading days of each week should get their own probability density, no matter on which day of the week they fall. *Before increasing the complexity of a model, you should investigate whether the costs of doing so outweigh the benefits.* Specifically, you should investigate whether or not there is benefit in treating any one trading day of the week differently than the others before building a more complicated models.

Remark. IID models are also the simplest models that are consistent with the way any portfolio theory is used. Specifically, to use any portfolio theory you must first calibrate a model from historical data. This model is then used to predict how a set of ideal portfolios might behave in the future. Based on these predictions one selects the ideal portfolio that optimizes some objective. *This strategy makes the implicit assumption that in the future the market will behave statistically as it did in the past.*

This assumption requires the market statistics to be stable relative to its dynamics. But this requires future states to decorrelate from past states. Markov models are characterized by the assumption that possible future states are independent of past states, which maximizes this decorrelation. IID models are the simplest Markov models. All the models discussed in the previous remark are also Markov models. We will use only IID models.

Return Rate Probability Densities. Once you have decided to use an IID model for a particular asset, you might think the next goal is to pick an appropriate probability density $q(R)$. However, that is neither practical nor necessary. *Rather, the goal is to identify appropriate statistical information about $q(R)$ that sheds light on the market. Ideally this information should be insensitive to details of $q(R)$ within a large class of probability densities.* Statisticians call such an approach *nonparametric.*

The expected value of any function $\psi(R)$ is given by

$$\mathrm{Ex}\Big(\psi(R)\Big) = \int_{-D}^{\infty} \psi(R)\, q(R)\, \mathrm{d}R\,,$$

provided $|\psi(R)|\, q(R)$ is integrable. Because we have been collecting mean and covariance return rate data, we will assume that the probability densities satisfy

$$\int_{-D}^{\infty} R^2 q(R)\, \mathrm{d}R < \infty\,.$$

The mean μ and variance ξ of R are then

$$\mu = \mathsf{Ex}(R) = \int_{-D}^{\infty} R\, q(R)\, \mathrm{d}R\,,$$

$$\xi = \mathsf{Var}(R) = \mathsf{Ex}\Big((R-\mu)^2\Big) = \int_{-D}^{\infty} (R-\mu)^2\, q(R)\, \mathrm{d}R\,.$$

However we do not know these. Rather, we must infer them from the data, at least approximately. Given D_h samples $\{R(d)\}_{d=1}^{D_h}$ that are drawn from the density $q(R)$, we can construct an *estimator* $\widehat{\mu}$ of μ by

$$\widehat{\mu} = \sum_{d=1}^{D_h} w(d)\, R(d)\,.$$

This is so-called *sample mean* is an *unbiased estimator* of μ because

$$\mathsf{Ex}(\widehat{\mu}) = \sum_{d=1}^{D_h} w(d)\, \mathsf{Ex}\Big(R(d)\Big) = \sum_{d=1}^{D_h} w(d)\, \mu = \mu\,.$$

We can estimate how close $\widehat{\mu}$ is to μ by computing its variance as

$$\begin{aligned}
\mathsf{Var}(\widehat{\mu}) &= \mathsf{Ex}\Big((\widehat{\mu}-\mu)^2\Big) \\
&= \mathsf{Ex}\left(\sum_{d=1}^{D_h}\sum_{d'=1}^{D_h} w(d)\,w(d')\,(R(d)-\mu)\,(R(d')-\mu)\right) \\
&= \sum_{d=1}^{D_h}\sum_{d'=1}^{D_h} w(d)\,w(d')\,\mathsf{Ex}\Big((R(d)-\mu)\,(R(d')-\mu)\Big) \\
&= \sum_{d=1}^{D_h} w(d)^2\mathsf{Ex}\Big((R(d)-\mu)^2\Big) = \sum_{d=1}^{D_h} w(d)^2\xi = \bar{w}\,\xi\,.
\end{aligned}$$

Here the off-diagonal terms in the double sum vanish because

$$\mathsf{Ex}\Big((R(d)-\mu)\,(R(d')-\mu)\Big) = 0 \quad \text{when } d \neq d'\,.$$

The fact $\mathsf{Var}(\widehat{\mu}) = \bar{w}\xi$ implies that $\widehat{\mu}$ converges to μ like $\sqrt{\bar{w}}$ as $D_h \to \infty$. This rate is fastest for uniform weights, when it is $1/\sqrt{D_h}$ as $D_h \to \infty$.

We can construct an *unbiased estimator* of ξ that is proportional to the so-called *sample variance* as

$$\widehat{\xi} = \frac{1}{1-\bar{w}} \sum_{d=1}^{D_h} w(d)\Big(R(d) - \widehat{\mu}\Big)^2 .$$

Indeed, from the calculation on the previous slide we confirm that

$$\begin{aligned}
\mathsf{Ex}\big(\widehat{\xi}\big) &= \frac{1}{1-\bar{w}}\,\mathsf{Ex}\left(\sum_{d=1}^{D_h} w(d)\Big(R(d)-\mu\Big)^2 - (\widehat{\mu}-\mu)^2\right) \\
&= \sum_{d=1}^{D_h} \frac{w(d)}{1-\bar{w}}\,\mathsf{Ex}\Big(\Big(R(d)-\mu\Big)^2\Big) - \frac{\mathsf{Ex}\big((\widehat{\mu}-\mu)^2\big)}{1-\bar{w}} \\
&= \sum_{d=1}^{D_h} \frac{w(d)}{1-\bar{w}}\,\xi - \frac{\bar{w}\,\xi}{1-\bar{w}} = \frac{\xi}{1-\bar{w}} - \frac{\bar{w}\,\xi}{1-\bar{w}} = \xi .
\end{aligned}$$

Remark. *The factor $1/(1-\bar{w})$ in $\widehat{\xi}$ is the same factor that appears in $\mathbf{V}$.*

Growth Rate Probability Densities. Given D_h samples $\{R(d)\}_{d=1}^{D_h}$ that are drawn from the return rate probability density $q(R)$, the associated simulated share prices satisfy

$$S(d) = \left(1 + \tfrac{1}{D} R(d)\right) S(d-1)\,, \quad \text{for } d = 1, \cdots, D_h\,.$$

If we set $S(0) = s(0)$ then you can easily see that

$$S(d) = \prod_{d'=1}^{d} \left(1 + \tfrac{1}{D} R(d')\right) s(0)\,.$$

The *growth rate* $X(d)$ is related to the return rate $R(d)$ by

$$e^{\frac{1}{D} X(d)} = 1 + \tfrac{1}{D} R(d)\,.$$

In other words, $X(d)$ is the growth rate that yeilds a return rate $R(d)$ on trading day d. The formula for $S(d)$ then takes the form

$$S(d) = \exp\left(\frac{1}{D} \sum_{d'=1}^{d} X(d')\right) s(0)\,.$$

When $\{R(d)\}_{d=1}^{D_h}$ is an IID process drawn from the density $q(R)$ over $(-D,\infty)$, it follows that $\{X(d)\}_{d=1}^{D_h}$ is an IID process drawn from the density $p(X)$ over $(-\infty,\infty)$ where $p(X)\,\mathrm{d}X = q(R)\,\mathrm{d}R$ with X and R related by

$$X = D \log\!\left(1 + \tfrac{1}{D}R\right) , \qquad R = D\left(e^{\frac{1}{D}X} - 1\right) .$$

More explicitly, the densities $p(X)$ and $q(R)$ are related by

$$p(X) = q\!\left(D\left(e^{\frac{1}{D}X} - 1\right)\right) e^{\frac{1}{D}X} , \qquad q(R) = \frac{p\!\left(D\log\!\left(1+\frac{1}{D}R\right)\right)}{1+\frac{1}{D}R} .$$

Because our models will involve means and variances, we will require that

$$\int_{-\infty}^{\infty} X^2 p(X)\,\mathrm{d}X = \int_{-D}^{\infty} D^2 \log\!\left(1+\tfrac{1}{D}R\right)^2 q(R)\,\mathrm{d}R < \infty ,$$
$$\int_{-\infty}^{\infty} D^2\left(e^{\frac{1}{D}X} - 1\right)^2 p(X)\,\mathrm{d}X = \int_{-D}^{\infty} R^2 q(R)\,\mathrm{d}R < \infty .$$

The big advantage of working with $p(X)$ rather than $q(R)$ is the fact that

$$\log\left(\frac{S(d)}{s(0)}\right) = \frac{1}{D}\sum_{d'=1}^{d} X(d') \,.$$

In other words, $\log(S(d)/s(0))$ is a sum of an IID process. It is easy to compute the mean and variance of this quantity in terms of those of X.

The mean γ and variance θ of X are

$$\gamma = \mathrm{Ex}(X) = \int_{-\infty}^{\infty} X\, p(X)\, \mathrm{d}X \,,$$

$$\theta = \mathrm{Var}(X) = \mathrm{Ex}\Big((X-\gamma)^2\Big) = \int_{-\infty}^{\infty} (X-\gamma)^2\, p(X)\, \mathrm{d}X \,.$$

For the mean of $\log(S(d)/s(0))$ we find that

$$\mathrm{Ex}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \frac{1}{D}\sum_{d'=1}^{d} \mathrm{Ex}\Big(X(d')\Big) = \tfrac{d}{D}\,\gamma \,,$$

For the variance of $\log(S(d)/s(0))$ we find that

$$\begin{aligned}
\mathrm{Var}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) &= \mathrm{Ex}\left(\left(\frac{1}{D}\sum_{d'=1}^{d} X(d') - \tfrac{d}{D}\,\gamma\right)^2\right) \\
&= \frac{1}{D^2}\,\mathrm{Ex}\left(\left(\sum_{d'=1}^{d}\left(X(d') - \gamma\right)\right)^2\right) \\
&= \frac{1}{D^2}\,\mathrm{Ex}\left(\sum_{d'=1}^{d}\sum_{d''=1}^{d}\left(X(d') - \gamma\right)\left(X(d'') - \gamma\right)\right) \\
&= \frac{1}{D^2}\sum_{d'=1}^{d}\mathrm{Ex}\left(\left(X(d') - \gamma\right)^2\right) = \frac{d}{D^2}\,\theta\,.
\end{aligned}$$

Here the off-diagonal terms in the double sum vanish because

$$\mathrm{Ex}\left(\left(X(d') - \gamma\right)\left(X(d'') - \gamma\right)\right) = 0 \qquad \text{when } d'' \neq d'\,.$$

Therefore the expected growth and variance of the IID model asset at time $t = d/D$ years is

$$\mathsf{Ex}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \gamma\, t\,, \qquad \mathsf{Var}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \tfrac{1}{D}\,\theta\, t\,.$$

Remark. *The IID model suggests that the growth rate mean γ is a good proxy for the reward of an asset and that $\sqrt{\frac{1}{D}\,\theta}$ is a good proxy for its risk. However, these are not the proxies chosen by MPT when it is applied to a portfolio consisting of one risky asset.* These proxies can be approximated by $\widehat{\gamma}$ and $\sqrt{\frac{1}{D}\,\widehat{\theta}}$ where $\widehat{\gamma}$ and $\widehat{\theta}$ are the unbiased estimators of γ and θ given by

$$\widehat{\gamma} = \sum_{d=1}^{D_h} w(d)\, X(d)\,, \qquad \widehat{\theta} = \sum_{d=1}^{D_h} \frac{w(d)}{1-\bar{w}}\Big(X(d) - \widehat{\gamma}\Big)^2\,.$$

Normal Growth Rate Model. We can illustrate what is going on with the simple IID model where $p(X)$ is the *normal* or *Gaussian* density with mean γ and variance θ, which is given by

$$p(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) .$$

Let $\{X(d)\}_{d=1}^{\infty}$ be a sequence of IID random variables drawn from $p(X)$. Let $\{Y(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y(d) = \frac{1}{d}\sum_{d'=1}^{d} X(d') \qquad \text{for every } d = 1, \cdots, \infty .$$

You can easily check that

$$\mathrm{Ex}(Y(d)) = \gamma , \qquad \mathrm{Var}(Y(d)) = \frac{\theta}{d} .$$

You can also check that $\mathrm{Ex}(Y(d)|Y(d-1)) = \frac{d-1}{d}Y(d-1) + \frac{1}{d}\gamma$. So the variables $Y(d)$ are neither independent nor identically distributed.

It can be shown (the details are not given here) that $Y(d)$ is drawn from the normal density with mean γ and variance θ/d, which is given by

$$p_d(Y) = \sqrt{\frac{d}{2\pi\theta}}\, \exp\!\left(-\frac{(Y-\gamma)^2 d}{2\theta}\right) .$$

Because $S(d)/s(0) = e^{\frac{d}{D}Y(d)}$, the mean return at day d is

$$\begin{aligned}
\mathrm{Ex}\!\left(e^{\frac{d}{D}Y(d)}\right) &= \sqrt{\frac{d}{2\pi\theta}} \int \exp\!\left(-\frac{(Y-\gamma)^2 d}{2\theta} + \tfrac{d}{D}Y\right) \mathrm{d}Y \\
&= \sqrt{\frac{d}{2\pi\theta}} \int \exp\!\left(-\frac{(Y-\gamma-\frac{1}{D}\theta)^2 d}{2\theta} + \tfrac{d}{D}(\gamma + \tfrac{1}{2D}\theta)\right) \mathrm{d}Y \\
&= \exp\!\left(\tfrac{d}{D}(\gamma + \tfrac{1}{2D}\theta)\right) .
\end{aligned}$$

This grows at rate $\gamma + \frac{1}{2D}\theta$, which is higher than the rate γ that most investors see. Indeed, we see that $p_d(Y)$ becomes more sharply peaked around $Y = \gamma$ as d increases.

By setting $d = 1$ in the above formula, we see that the return rate mean is

$$\mu = \mathsf{Ex}(R) = D\,\mathsf{Ex}\Big(e^{\frac{1}{D}X} - 1\Big) = D\,\Big(\exp\!\big(\tfrac{1}{D}(\gamma + \tfrac{1}{2D}\theta)\big) - 1\Big)\,.$$

Therefore $\mu > \gamma + \frac{1}{2D}\theta$, with $\mu \approx \gamma + \frac{1}{2D}\theta$ when $\frac{1}{D}(\gamma + \frac{1}{2D}\theta) << 1$. *This shows that most investors will see a return rate that is below the return rate mean μ — far below in volatile markets.* This is because $e^{\frac{1}{D}X}$ amplifies the tail of the normal density. For a more realistic IID model with a density $p(X)$ that decays more slowly than a normal density as $X \to \infty$, this difference can be more striking. Said another way, most investors will not see the same return as Warren Buffett, but his return will boost the mean.

The normal growth rate model confirms that γ is a better proxy for how well a risky asset might perform than μ because $p_d(Y)$ becomes more peaked around $Y = \gamma$ as d increases. We will extend this result to a general class of IID models that are more realistic.

Exercise. Use the unbiased estimators $\widehat{\mu}$, $\widehat{\xi}$, $\widehat{\gamma}$, and $\widehat{\theta}$ given by

$$\widehat{\mu} = \frac{1}{D} \sum_{d=1}^{D} r(d) \,, \qquad \widehat{\xi} = \frac{1}{D-1} \sum_{d=1}^{D} \Big(r(d) - \widehat{\mu} \Big)^2 \,,$$

$$\widehat{\gamma} = \frac{1}{D} \sum_{d=1}^{D} x(d) \,, \qquad \widehat{\theta} = \frac{1}{D-1} \sum_{d=1}^{D} \Big(x(d) - \widehat{\gamma} \Big)^2 \,,$$

to estimate μ, ξ, γ, and θ given the share price history $\{s(d)\}_{d=0}^{D}$ with

$$r(d) = D \left(\frac{s(d)}{s(d-1)} - 1 \right) \,, \qquad x(d) = D \log\left(\frac{s(d)}{s(d-1)} \right) \,,$$

for each of the following assets. How do $\widehat{\mu}$ and $\widehat{\gamma}$ compare? $\widehat{\xi}$ and $\widehat{\theta}$?

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.