Modeling Portfolios that Contain Risky Assets

Portfolio Models I: Portfolios with Risk-Free Assets

C. David Levermore
University of Maryland, College Park

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Portfolio Models I: Portfolios with Risk-Free Assets

Risk-Free Assets. Until now we have considered portfolios that contain only risky assets. We now consider two kinds of *risk-free* assets (assets that have no volatility associated with them) that can play a major role in portfolio management.

The first is a *safe investment* that pays dividends at a prescribed interest rate μ_{si}. This can be an FDIC insured bank account, or safe securities such as US Treasury Bills, Notes, or Bonds. (U.S. Treasury Bills are used most commonly.) *You can only hold a long position in such an asset.*

The second is a *credit line* from which you can borrow at a prescribed interest rate μ_{cl} up to your credit limit. Such a credit line should require you to put up assets like real estate or part of your portfolio (a *margin*) as collateral from which the borrowed money can be recovered if need be. *You can only hold a short position in such an asset.*

We will assume that $\mu_{\text{cl}} \geq \mu_{\text{si}}$, because otherwise investors would make money by borrowing at rate μ_{cl} in order to invest at the greater rate μ_{si}. (Here we are again neglecting transaction costs.) Because free money does not sit around for long, market forces would quickly adjust the rates so that $\mu_{\text{cl}} \geq \mu_{\text{si}}$. In practice, μ_{cl} is about three points higher than μ_{si}.

We will also assume that a portfolio will not hold a position in both the safe investment and the credit line when $\mu_{\text{cl}} > \mu_{\text{si}}$. To do so would effectively be borrowing at rate μ_{cl} in order to invest at the lesser rate μ_{si}. While there can be cash-flow management reasons for holding such a position for a short time, it is not a smart long term position.

These assumptions imply that every portfolio can be viewed as holding a position in at most one risk-free asset: it can hold either a long position at rate μ_{si}, a short position at rate μ_{cl}, or a neutral risk-free position.

Markowitz Portfolios. We now extend the notion of Markowitz portfolios to portfolios that might include a single risk-free asset with return rate μ_{rf}. Let $b_{\mathrm{rf}}(d)$ denote the balance in the risk-free asset at the start of day d. For a long position $\mu_{\mathrm{rf}} = \mu_{\mathrm{si}}$ and $b_{\mathrm{rf}}(d) > 0$, while for a short position $\mu_{\mathrm{rf}} = \mu_{\mathrm{cl}}$ and $b_{\mathrm{rf}}(d) < 0$.

A *Markowitz portfolio* containing one risk-free asset and N risky assets is uniquely determined by real numbers f_{rf} and $\{f_i\}_{i=1}^N$ that satisfy

$$f_{\mathrm{rf}} + \sum_{i=1}^{N} f_i = 1\,, \qquad f_{\mathrm{rf}} < 1 \quad \text{if any } f_j \neq 0\,.$$

The portfolio is rebalanced at the start of each day so that

$$\frac{b_{\mathrm{rf}}(d)}{\Pi(d-1)} = f_{\mathrm{rf}}\,, \qquad \frac{n_i(d)\, s_i(d-1)}{\Pi(d-1)} = f_i \qquad \text{for } i = 1, \cdots, N\,.$$

The condition $f_{\mathrm{rf}} < 1$ if any $f_j \neq 0$ states that the safe investment must contain less than the net portfolio value unless it is the entire portfolio.

Its value at the start of day d is

$$\Pi(d-1) = b_{\mathsf{rf}}(d) + \sum_{i=1}^{N} n_i(d)\, s_i(d-1)\,,$$

while its value at the end of day d is

$$\Pi(d) = b_{\mathsf{rf}}(d)\left(1 + \tfrac{1}{D}\,\mu_{\mathsf{rf}}\right) + \sum_{i=1}^{N} n_i(d)\, s_i(d)\,.$$

Its return rate for day d is therefore

$$\begin{aligned} r(d) &= D\,\frac{\Pi(d) - \Pi(d-1)}{\Pi(d-1)} \\ &= \frac{b_{\mathsf{rf}}(d)\,\mu_{\mathsf{rf}}}{\Pi(d-1)} + \sum_{i=1}^{N} D\,\frac{n_i(d)\Big(s_i(d) - s_i(d-1)\Big)}{\Pi(d-1)} \\ &= f_{\mathsf{rf}}\,\mu_{\mathsf{rf}} + \sum_{i=1}^{N} f_i\, r_i(d) = f_{\mathsf{rf}}\,\mu_{\mathsf{rf}} + \mathbf{f}^{\mathsf{T}}\mathbf{r}(d)\,. \end{aligned}$$

The portfolio return rate mean μ and variance v are then given by

$$\begin{aligned}
\mu &= \sum_{d=1}^{D_h} w(d)\, r(d) = \sum_{d=1}^{D_h} w(d) \left(f_{\mathsf{rf}}\, \mu_{\mathsf{rf}} + \mathbf{f}^{\mathsf{T}} \mathbf{r}(d) \right) \\
&= f_{\mathsf{rf}}\, \mu_{\mathsf{rf}} + \mathbf{f}^{\mathsf{T}} \left(\sum_{d=1}^{D_h} w(d)\, \mathbf{r}(d) \right) = f_{\mathsf{rf}}\, \mu_{\mathsf{rf}} + \mathbf{f}^{\mathsf{T}} \mathbf{m}\,, \\
v &= \tfrac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1-\bar{w}} \left(r(d) - \mu \right)^2 = \tfrac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1-\bar{w}} \left(\mathbf{f}^{\mathsf{T}} \mathbf{r}(d) - \mathbf{f}^{\mathsf{T}} \mathbf{m} \right)^2 \\
&= \mathbf{f}^{\mathsf{T}} \left(\tfrac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1-\bar{w}} \left(\mathbf{r}(d) - \mathbf{m} \right) \left(\mathbf{r}(d) - \mathbf{m} \right)^{\mathsf{T}} \right) \mathbf{f} = \mathbf{f}^{\mathsf{T}} \mathbf{V} \mathbf{f}\,.
\end{aligned}$$

We thereby obtain the formulas

$$\mu = \mu_{\mathsf{rf}} \left(1 - \mathbf{1}^{\mathsf{T}} \mathbf{f} \right) + \mathbf{m}^{\mathsf{T}} \mathbf{f}\,, \qquad v = \mathbf{f}^{\mathsf{T}} \mathbf{V} \mathbf{f}\,.$$

Capital Allocation Lines. These formulas can be viewed as describing a point that lies on a certain half-line in the $\sigma\mu$-plane. Let (σ, μ) be the point in the $\sigma\mu$-plane associated with the Markowitz portfolio characterized by the distribution $\mathbf{f} \neq \mathbf{0}$. Notice that $\mathbf{1}^\mathsf{T}\mathbf{f} = 1 - f_{\mathsf{rf}} > 0$ because $\mathbf{f} \neq \mathbf{0}$. Define

$$\tilde{\mathbf{f}} = \frac{\mathbf{f}}{\mathbf{1}^\mathsf{T}\mathbf{f}}\,.$$

Notice that $\mathbf{1}^\mathsf{T}\tilde{\mathbf{f}} = 1$. Let $\tilde{\mu} = \mathbf{m}^\mathsf{T}\tilde{\mathbf{f}}$ and $\tilde{\sigma} = \sqrt{\tilde{\mathbf{f}}^\mathsf{T}\mathbf{V}\tilde{\mathbf{f}}}$. Then $(\tilde{\sigma}, \tilde{\mu})$ is the point in the $\sigma\mu$-plane associated with the Markowitz portfolio without risk-free assets that is characterized by the distribution $\tilde{\mathbf{f}}$. Because

$$\mu = \left(1 - \mathbf{1}^\mathsf{T}\mathbf{f}\right)\mu_{\mathsf{rf}} + \mathbf{1}^\mathsf{T}\mathbf{f}\,\tilde{\mu}\,, \qquad \sigma = \mathbf{1}^\mathsf{T}\mathbf{f}\,\tilde{\sigma}\,,$$

we see that the point (σ, μ) in the $\sigma\mu$-plane lies on the half-line that starts at the point $(0, \mu_{\mathsf{rf}})$ and passes through the point $(\tilde{\sigma}, \tilde{\mu})$ that corresponds to a portfolio that does not contain the risk-free asset.

Conversely, given any point $(\tilde{\sigma}, \tilde{\mu})$ corresponding to a Markowitz portfolio that contains no risk-free assets, consider the half-line

$$(\sigma, \mu) = \Big(\phi\,\tilde{\sigma}\,,\ (1-\phi)\mu_{\mathsf{rf}} + \phi\,\tilde{\mu}\Big) \qquad \text{where } \phi > 0\,.$$

If a portfolio corresponding to $(\tilde{\sigma}, \tilde{\mu})$ has distribution $\tilde{\mathbf{f}}$ then the point on the half-line given by ϕ corresponds to the portfolio with distribution $\mathbf{f} = \phi\tilde{\mathbf{f}}$. This portfolio allocates $1 - \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1 - \phi$ of its value to the risk-free asset. The risk-free asset is held long if $\phi \in (0, 1)$ and held short if $\phi > 1$ while $\phi = 1$ corresponds to a neutral position. We must restrict ϕ to either $(0, 1]$ or $[1, \infty)$ depending on whether the risk-free asset is the safe investment or the credit line. This segment of the half-line is called the *capital allocation line* through $(\tilde{\sigma}, \tilde{\mu})$ associated with the risk-free asset.

We can therefore use the appropriate capital allocation lines to construct the set of all points in the $\sigma\mu$-plane associated with Markowitz portfolios that contain a risk-free asset from the set of all points in the $\sigma\mu$-plane associated with Markowitz portfolios that contain no risk-free assets.

Efficient Frontier. We now use the capital allocation line construction to see how the efficient frontier is modified by including risk-free assets. Recall that the efficient frontier for portfolios that contain no risk-free assets is given by

$$\mu = \mu_{\mathsf{mv}} + \nu_{\mathsf{as}}\sqrt{\sigma^2 - \sigma_{\mathsf{mv}}^2} \qquad \text{for } \sigma \geq \sigma_{\mathsf{mv}}\,.$$

Every point $(\tilde{\sigma}, \tilde{\mu})$ on this curve has a unique frontier portfolio associated with it. Because $\mu_{\mathsf{rf}} < \mu_{\mathsf{mv}}$ there is a unique half-line that starts at the point $(0, \mu_{\mathsf{rf}})$ and is tangent to this curve. Denote this half-line by

$$\mu = \mu_{\mathsf{rf}} + \nu_{\mathsf{tg}}\,\sigma \qquad \text{for } \sigma \geq 0\,.$$

Let $(\sigma_{\mathsf{tg}}, \mu_{\mathsf{tg}})$ be the point at which this tangency occurs. The unique frontier portfolio associated with this point is called the *tangency portfolio* associated with the risk-free asset; it has distribution $\mathbf{f}_{\mathsf{tg}} = \mathbf{f}_{\mathsf{f}}(\mu_{\mathsf{tg}})$. Then the appropriate capital allocation line will be part of the efficient frontier.

The so-called *tangency parameters*, σ_{tg}, μ_{tg}, and ν_{tg}, can be determined from the equations

$$\mu_{\text{tg}} = \mu_{\text{rf}} + \nu_{\text{tg}}\,\sigma_{\text{tg}}\,, \qquad \mu_{\text{tg}} = \mu_{\text{mv}} + \nu_{\text{as}}\sqrt{\sigma_{\text{tg}}^2 - \sigma_{\text{mv}}^2}\,,$$

$$\nu_{\text{tg}} = \frac{\nu_{\text{as}}\,\sigma_{\text{tg}}}{\sqrt{\sigma_{\text{tg}}^2 - \sigma_{\text{mv}}^2}}\,.$$

The first equation states that $(\sigma_{\text{tg}}, \mu_{\text{tg}})$ lies on the capital allocation line. The second states that it also lies on the efficient frontier curve for portfolios that contain no risk-free assets. The third equates the slope of the capital allocation line to that of the efficient frontier curve at the point $(\sigma_{\text{tg}}, \mu_{\text{tg}})$. By using the last equation to eliminate ν_{tg} from the first, and then using the resulting equation to eliminate μ_{tg} from the second, we find that

$$\frac{\mu_{\text{mv}} - \mu_{\text{rf}}}{\nu_{\text{as}}} = \frac{\sigma_{\text{tg}}^2}{\sqrt{\sigma_{\text{tg}}^2 - \sigma_{\text{mv}}^2}} - \sqrt{\sigma_{\text{tg}}^2 - \sigma_{\text{mv}}^2} = \frac{\sigma_{\text{mv}}^2}{\sqrt{\sigma_{\text{tg}}^2 - \sigma_{\text{mv}}^2}}\,.$$

We thereby obtain

$$\sigma_{\mathsf{tg}} = \sigma_{\mathsf{mv}}\sqrt{1 + \left(\frac{\nu_{\mathsf{as}}\,\sigma_{\mathsf{mv}}}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}\right)^2}\,, \qquad \mu_{\mathsf{tg}} = \mu_{\mathsf{mv}} + \frac{\nu_{\mathsf{as}}^2\,\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}\,,$$

$$\nu_{\mathsf{tg}} = \nu_{\mathsf{as}}\sqrt{1 + \left(\frac{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}{\nu_{\mathsf{as}}\,\sigma_{\mathsf{mv}}}\right)^2}\,.$$

The distribution $\mathbf{f}_{\mathsf{tg}}$ of the tangency portfolio is then given by

$$\begin{aligned}
\mathbf{f}_{\mathsf{tg}} = \mathbf{f}_{\mathsf{f}}(\mu_{\mathsf{tg}}) &= \mathbf{f}_{\mathsf{mv}} + \frac{\mu_{\mathsf{tg}} - \mu_{\mathsf{mv}}}{\nu_{\mathsf{as}}^2}\,\mathbf{V}^{-1}\big(\mathbf{m} - \mu_{\mathsf{mv}}\mathbf{1}\big) \\
&= \sigma_{\mathsf{mv}}^2\mathbf{V}^{-1}\mathbf{1} + \frac{\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}\,\mathbf{V}^{-1}\big(\mathbf{m} - \mu_{\mathsf{mv}}\mathbf{1}\big) \\
&= \frac{\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}\,\mathbf{V}^{-1}\big(\mathbf{m} - \mu_{\mathsf{rf}}\mathbf{1}\big)\,.
\end{aligned}$$

Remark. The above formulas can be applied to either the safe investment or the credit line by simply choosing the appropriate value of μ_{rf}.

Example: One Risk-Free Rate Model. First consider the case when $\mu_{\text{si}} = \mu_{\text{cl}} < \mu_{\text{mv}}$. Set $\mu_{\text{rf}} = \mu_{\text{si}} = \mu_{\text{cl}}$ and let ν_{tg} and σ_{tg} be the slope and volatility of the tangency portfolio that is common to both the safe investment and the credit line. The efficient frontier is then given by

$$\mu_{\text{ef}}(\sigma) = \mu_{\text{rf}} + \nu_{\text{tg}}\,\sigma \qquad \text{for } \sigma \in [0,\infty)\,.$$

Let $\mathbf{f}_{\text{tg}}$ be the distribution of the tangency portfolio that is common to both the safe investment and the credit line. The distribution of the associated portfolio is then given by

$$\mathbf{f}_{\text{ef}}(\sigma) = \frac{\sigma}{\sigma_{\text{tg}}}\,\mathbf{f}_{\text{tg}} \qquad \text{for } \sigma \in [0,\infty)\,,$$

These portfolios are constructed as follows:

1. if $\sigma = 0$ then the investor holds only the safe investment;

2. if $\sigma \in (0, \sigma_{\mathrm{tg}})$ then the investor places

   - $\dfrac{\sigma_{\mathrm{tg}} - \sigma}{\sigma_{\mathrm{tg}}}$ of the portfolio value in the safe investment,
   - $\dfrac{\sigma}{\sigma_{\mathrm{tg}}}$ of the portfolio value in the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$;

3. if $\sigma = \sigma_{\mathrm{tg}}$ then the investor holds only the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$;

4. if $\sigma \in (\sigma_{\mathrm{tg}}, \infty)$ then the investor places

   - $\dfrac{\sigma}{\sigma_{\mathrm{tg}}}$ of the portfolio value in the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$,
   - by borrowing $\dfrac{\sigma - \sigma_{\mathrm{tg}}}{\sigma_{\mathrm{tg}}}$ of this value from the credit line.

Example: Two Risk-Free Rates Model. Next consider the case when $\mu_{\text{si}} < \mu_{\text{cl}} < \mu_{\text{mv}}$. Let ν_{st} and σ_{st} be the slope and volatility of the so-called *safe tangency portfolio* associated with the safe investment. Let ν_{ct} and σ_{ct} be the slope and volatility of the so-called *credit tangency portfolio* associated with the credit line. The efficient frontier is then given by

$$\mu_{\text{ef}}(\sigma) = \begin{cases} \mu_{\text{si}} + \nu_{\text{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\text{st}}]\,, \\ \mu_{\text{mv}} + \nu_{\text{as}}\sqrt{\sigma^2 - \sigma_{\text{mv}}^2} & \text{for } \sigma \in [\sigma_{\text{st}}, \sigma_{\text{ct}}]\,, \\ \mu_{\text{cl}} + \nu_{\text{ct}}\,\sigma & \text{for } \sigma \in [\sigma_{\text{ct}}, \infty)\,, \end{cases}$$

where

$$\nu_{\text{st}} = \nu_{\text{as}}\sqrt{1 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{si}}}{\nu_{\text{as}}\,\sigma_{\text{mv}}}\right)^2}\,, \qquad \sigma_{\text{st}} = \sigma_{\text{mv}}\sqrt{1 + \left(\frac{\nu_{\text{as}}\,\sigma_{\text{mv}}}{\mu_{\text{mv}} - \mu_{\text{si}}}\right)^2}\,,$$

$$\nu_{\text{ct}} = \nu_{\text{as}}\sqrt{1 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{cl}}}{\nu_{\text{as}}\,\sigma_{\text{mv}}}\right)^2}\,, \qquad \sigma_{\text{ct}} = \sigma_{\text{mv}}\sqrt{1 + \left(\frac{\nu_{\text{as}}\,\sigma_{\text{mv}}}{\mu_{\text{mv}} - \mu_{\text{cl}}}\right)^2}\,.$$

The return rate means for the tangency portfolios are

$$\mu_{\text{st}} = \mu_{\text{mv}} + \frac{\nu_{\text{as}}^2\,\sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{si}}}\,, \qquad \mu_{\text{ct}} = \mu_{\text{mv}} + \frac{\nu_{\text{as}}^2\,\sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{cl}}}\,,$$

while the distributions of risky assets for these portfolios are

$$\mathbf{f}_{\text{st}} = \frac{\sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{si}}}\,\mathbf{V}^{-1}\Big(\mathbf{m} - \mu_{\text{si}}\mathbf{1}\Big)\,, \quad \mathbf{f}_{\text{ct}} = \frac{\sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{cl}}}\,\mathbf{V}^{-1}\Big(\mathbf{m} - \mu_{\text{cl}}\mathbf{1}\Big)\,.$$

By the two fund property, the distribution of risky assets for any efficient frontier portfolio is then given by

$$\mathbf{f}_{\text{ef}}(\sigma) = \begin{cases} \dfrac{\sigma}{\sigma_{\text{st}}}\,\mathbf{f}_{\text{st}} & \text{for } \sigma \in [0, \sigma_{\text{st}}]\,, \\ \dfrac{\mu_{\text{ct}} - \mu_{\text{ef}}(\sigma)}{\mu_{\text{ct}} - \mu_{\text{st}}}\,\mathbf{f}_{\text{st}} + \dfrac{\mu_{\text{ef}}(\sigma) - \mu_{\text{st}}}{\mu_{\text{ct}} - \mu_{\text{st}}}\,\mathbf{f}_{\text{ct}} & \text{for } \sigma \in (\sigma_{\text{st}}, \sigma_{\text{ct}})\,, \\ \dfrac{\sigma}{\sigma_{\text{ct}}}\,\mathbf{f}_{\text{ct}} & \text{for } \sigma \in [\sigma_{\text{ct}}, \infty)\,. \end{cases}$$

These portfolios are constructed as follows:

1. if $\sigma = 0$ then the investor holds only the safe investment;

2. if $\sigma \in (0, \sigma_{\text{st}})$ then the investor places

   - $\dfrac{\sigma_{\text{st}} - \sigma}{\sigma_{\text{st}}}$ of the portfolio value in the safe investment,
   - $\dfrac{\sigma}{\sigma_{\text{st}}}$ of the portfolio value in the safe tangency portfolio $\mathbf{f}_{\text{st}}$;

3. if $\sigma = \sigma_{\text{st}}$ then the investor holds only the safe tangency portfolio $\mathbf{f}_{\text{st}}$;

4. if $\sigma \in (\sigma_{\text{st}}, \sigma_{\text{ct}})$ then the investor holds

- $\dfrac{\mu_{\text{ct}} - \mu_{\text{ef}}(\sigma)}{\mu_{\text{ct}} - \mu_{\text{st}}}$ of the portfolio value in the safe tangency portfolio $\mathbf{f}_{\text{st}}$,
- $\dfrac{\mu_{\text{ef}}(\sigma) - \mu_{\text{st}}}{\mu_{\text{ct}} - \mu_{\text{st}}}$ of the portfolio value in the credit tangency portfolio $\mathbf{f}_{\text{ct}}$;

5. if $\sigma = \sigma_{\text{ct}}$ then the investor holds only the credit tangency portfolio $\mathbf{f}_{\text{ct}}$;

6. if $\sigma \in (\sigma_{\text{ct}}, \infty)$ then the investor places

   - $\dfrac{\sigma}{\sigma_{\text{ct}}}$ of the portfolio value in the credit tangency portfolio $\mathbf{f}_{\text{ct}}$,
   - by borrowing $\dfrac{\sigma - \sigma_{\text{ct}}}{\sigma_{\text{ct}}}$ of this value from the credit line.

Remark. Some brokers will simply ask investors to select a σ that reflects the risk they are willing to take and then will build a portfolio for them that is near the efficient frontier portfolio $\mathbf{f}_{\mathrm{ef}}(\sigma)$. To guide this selection of σ, the broker will describe certain choices of σ as being "very conservative", "conservative", "aggressive", or "very aggressive" without quantifying what these terms mean. The performance of the resulting portfolio will often be disappointing to those who selected a "very conservative" σ and painful to those who selected a "very aggressive" σ. We will see reasons for this when we develop model-based methods for selecting a σ.

Credit Limits. The One Risk-Free Rate and Two Risk-Free Rates models breakdown when $\mu_{\mathrm{cl}} \geq \mu_{\mathrm{mv}}$ because in that case the capital allocation line construction fails. *Moreover, both models become unrealistic for large values of σ, especially when μ_{cl} is close to μ_{mv}, because they require an investor to borrow or to take short positions without bound.* In practice such positions are restricted by *credit limits*.

If we assume that in each case the lender is the broker and the collateral is part of your portfolio then a simple model for credit limits is to restrict the total short position of the portfolio to be a fraction $\ell \geq 0$ of the net portfolio value. The value of ℓ will depend upon market conditions, but brokers will often allow $\ell > 1$ and seldom allow $\ell > 5$. This model becomes

$$\frac{|1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}| - (1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})}{2} + \frac{|\mathbf{f}| - \mathbf{1}^{\mathsf{T}}\mathbf{f}}{2} \leq \ell\,, \quad \text{where} \quad |\mathbf{f}| = \sum_{i=1}^{N} |f_i|\,.$$

This simplifies to the convex constraint

$$|1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}| + |\mathbf{f}| \leq 1 + 2\ell\,.$$

This additional constraint makes the minimization problem more difficult. It can be solved numerically by using *primal-dual algorithms* from *convex optimization*. Rather than studying this general problem, in the next section we will consider the simpler problem of restricting to portfolios that contain only long positions, which is equivalent to taking $\ell = 0$ above.

Exercise. Consider the following groups of assets:

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Assume that μ_{si} is the US Treasury Bill rate at the end of the given year, and that μ_{cl} is three percentage points higher. On a single graph plot the points (σ_i, m_i) for every asset in groups (a) and (c) along with the efficient frontiers for group (a), for group (c), and for groups (a) and (c) combined taking into account the risk-free assets. Do the same thing for groups (b) and (d) on a second graph. (Use daily data.) Comment on any relationships you see between the objects plotted on each graph.

Exercise. Find $\mathbf{f}_{\mathrm{st}}$ and $\mathbf{f}_{\mathrm{ct}}$ for the six asset groupings in the preceding exercise. Compare the analogous groupings in 2007 and 2009.