Modeling Portfolios that Contain Risky Assets

Risk and Reward II: Markowitz Portfolios

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Risk and Reward II: Markowitz Portfolios

A 1952 paper by Harry Markowitz had enormous influence on the theory and practice of portfolio management and financial engineering ever since. It presented his doctoral dissertation work at the Unversity of Chicago, for which he was awarded the Nobel Prize in Economics in 1990. It was the first work to quantify how diversifying a portfolio can reduce its risk without changing its expected rate of return. It did this because it was the first work to use the covariance between different assets in an essential way.

Portfolios. We will consider portfolios in which an investor can hold one of three positions with respect to any risky asset. The investor can:

- hold a *long position* by owning shares of the asset;

- hold a *short position* by selling borrowed shares of the asset;

- hold a *neutral position* by doing neither of the above.

In order to keep things simple, we will not consider derivative assets.

Remark. You hold a short position by borrowing shares of an asset from a lender (usually your broker) and selling them immediately. If the share price subsequently goes down then you can buy the same number of shares and give them to the lender, thereby paying off your loan and profiting by the price difference minus transaction costs. Of course, if the price goes up then your lender can force you either to increase your collateral or to pay off the loan by buying shares at this higher price, thereby taking a loss that might be larger than the original value of the shares.

The value of any portfolio that holds $n_i(d)$ shares of asset i at the end of trading day d is

$$\Pi(d) = \sum_{i=1}^{N} n_i(d) s_i(d) \,.$$

If you hold a long position in asset i then $n_i(d) > 0$. If you hold a short position in asset i then $n_i(d) < 0$. If you hold a neutral position in asset i then $n_i(d) = 0$. We will assume that $\Pi(d) > 0$ for every d.

Markowitz carried out his analysis on a class of idealized portfolios that are each characterized by a set of real numbers $\{f_i\}_{i=1}^{N}$ such that

$$\sum_{i=1}^{N} f_i = 1 \,.$$

The portfolio picks $n_i(d)$ at the beginning at each trading day d so that

$$\frac{n_i(d)s_i(d-1)}{\Pi(d-1)} = f_i \,,$$

where $n_i(d)$ need not be an integer. We call these *Markowitz portfolios.* The portfolio holds a long position in asset i if $f_i > 0$ and holds a short position if $f_i < 0$. If every f_i is nonnegative then f_i is the fraction of the portfolio's value held in asset i at the beginning of each day. A Markowitz portfolio will be *self-financing* if we neglect trading costs because

$$\sum_{i=1}^{N} n_i(d)\, s_i(d-1) = \Pi(d-1) \,.$$

Portfolio Return Rate. We see from the self-financing property and the relationship between $n_i(d)$ and f_i that the return rate $r(d)$ of a Markowitz portfolio for trading day d is

$$
\begin{aligned}
r(d) &= D\, \frac{\Pi(d) - \Pi(d-1)}{\Pi(d-1)} \\
&= \sum_{i=1}^{N} D\, \frac{n_i(d)s_i(d) - n_i(d)s_i(d-1)}{\Pi(d-1)} \\
&= \sum_{i=1}^{N} \frac{n_i(d)s_i(d-1)}{\Pi(d-1)}\, D\, \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \sum_{i=1}^{N} f_i\, r_i(d)\,.
\end{aligned}
$$

The return rate $r(d)$ for the Markowitz portfolio characterized by $\{f_i\}_{i=1}^{N}$ is therefore simply the linear combination with coefficients f_i of the $r_i(d)$. *This relationship makes the class of Markowitz portfolios easy to analyze. We will therefore use Markowitz portfolios to model real portfolios.*

This relationship can be expressed in the compact form

$$r(d) = \mathbf{f}^\mathsf{T} \mathbf{r}(d) \,,$$

where $\mathbf{f}$ and $\mathbf{r}(d)$ are the N-vectors defined by

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \,, \qquad \mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix} \,.$$

Remark. Markowitz portfolios are easy to analyze because $r(d) = \mathbf{f}^\mathsf{T} \mathbf{r}(d)$ where $\mathbf{f}$ is *independent* of d. In particular, next we will show that for the Markowitz portfolio characterized by $\mathbf{f}$ we can express its return rate mean μ and variance v simply in terms of $\mathbf{m}$, $\mathbf{V}$ and $\mathbf{f}$.

Portfolio Statistics. Recall that if we assign weights $\{w(d)\}_{d=1}^{D_h}$ to the trading days of a return rate history $\{\mathbf{r}(d)\}_{d=1}^{D_h}$ then the N-vector of return rate means $\mathbf{m}$ and the $N \times N$-matrix of return rate covariances $\mathbf{V}$ can be expressed in terms of $\mathbf{r}(d)$ as

$$\mathbf{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix} = \sum_{d=1}^{D_h} w(d)\, \mathbf{r}(d)\,,$$

$$\mathbf{V} = \begin{pmatrix} v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{pmatrix} = \frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1-\bar{w}} \left(\mathbf{r}(d) - \mathbf{m}\right) \left(\mathbf{r}(d) - \mathbf{m}\right)^{\mathsf{T}}.$$

The choices of the return rate history $\{\mathbf{r}(d)\}_{d=1}^{D_h}$ and weights $\{w(d)\}_{d=1}^{D_h}$ specify the *calibration* of our models. *Ideally $\mathbf{m}$ and $\mathbf{V}$ should not be overly sensitive to these choices.*

Because $r(d) = \mathbf{f}^\top \mathbf{r}(d)$, the portfolio return rate mean μ and variance v for the Markowitz portfolio characterized by $\mathbf{f}$ are then given by

$$\mu = \sum_{d=1}^{D_h} w(d)\, r(d) = \sum_{d=1}^{D_h} w(d)\, \mathbf{f}^\top \mathbf{r}(d) = \mathbf{f}^\top \left(\sum_{d=1}^{D_h} w(d)\, \mathbf{r}(d) \right)$$

$$= \mathbf{f}^\top \mathbf{m}\,,$$

$$v = \frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} \left(r(d) - \mu \right)^2 = \frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} \left(\mathbf{f}^\top \mathbf{r}(d) - \mathbf{f}^\top \mathbf{m} \right)^2$$

$$= \frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} \left(\mathbf{f}^\top \mathbf{r}(d) - \mathbf{f}^\top \mathbf{m} \right)\left(\mathbf{r}(d)^\top \mathbf{f} - \mathbf{m}^\top \mathbf{f} \right)$$

$$= \mathbf{f}^\top \left(\frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} \left(\mathbf{r}(d) - \mathbf{m} \right)\left(\mathbf{r}(d) - \mathbf{m} \right)^\top \right) \mathbf{f} = \mathbf{f}^\top \mathbf{V} \mathbf{f}\,.$$

Hence, $\mu = \mathbf{f}^\top \mathbf{m}$ and $v = \mathbf{f}^\top \mathbf{V} \mathbf{f}$. Because $\mathbf{V}$ is positive definite, $v > 0$.

Remark. These simple formulas for μ and v are the reason that return rates are preferred over growth rates when compiling statistics of markets. The simplicity of these formulas arises because the return rates $r(d)$ for the Markowitz portfolio specified by the distribution $\mathbf{f}$ depends linearly upon the vector $\mathbf{r}(d)$ of return rates for the individual assets as $r(d) = \mathbf{f}^\mathsf{T} \mathbf{r}(d)$. In contrast, the growth rates $x(d)$ of a Markowitz portfolio are given by

$$
x(d) = D \log\left(\frac{\Pi(d)}{\Pi(d-1)}\right) = D \log\left(1 + \tfrac{1}{D} r(d)\right)
$$

$$
= D \log\left(1 + \tfrac{1}{D} \mathbf{f}^\mathsf{T} \mathbf{r}(d)\right) = D \log\left(1 + \tfrac{1}{D} \sum_{i=1}^{N} f_i\, r_i(d)\right)
$$

$$
= D \log\left(1 + \sum_{i=1}^{N} f_i \left(e^{\frac{1}{D} x_i(d)} - 1\right)\right) = D \log\left(\sum_{i=1}^{N} f_i\, e^{\frac{1}{D} x_i(d)}\right).
$$

Because the $x(d)$ are not linear functions of the $x_i(d)$, averages of $x(d)$ over d are not simply expressed in terms of averages of $x_i(d)$ over d.

Critique. Aspects of Markowitz portfolios are unrealistic. These include:

- the fact portfolios can contain fractional shares of any asset;
- the fact portfolios are rebalanced every trading day;
- the fact transaction costs and taxes are neglected;
- the fact dividends are neglected.

By making these simplifications the subsequent analysis becomes easier. The idea is to find the Markowitz portfolio that is best for a given investor. The expectation is that any real portfolio with a distribution close to that for the optimal Markowitz portfolio will perform nearly as well. Consequently, most investors rebalance at most a few times per year, and not every asset is involved each time. Transaction costs and taxes are thereby limited. Similarly, borrowing costs are kept to a minimum by not borrowing often. The model can be modified to account for dividends.

Remark. Portfolios of risky assets can be designed for *trading or investing.*

Traders often take positions that require constant attention. They might buy and sell assets on short time scales in an attempt to profit from market fluctuations. They might also take highly leveraged positions that expose them to enormous gains or loses depending how the market moves. They must be ready to handle margin calls. Trading is often a full time job.

Investors operate on longer time scales. They buy or sell an asset based on their assessment of its fundamental value over time. Investing does not have to be a full time job. Indeed, most people who hold risky assets are investors who are saving for retirement. Lured by the promise of high returns, sometimes investors will buy shares in funds designed for traders. At that point they have become gamblers, whether they realize it or not.

The ideas presented in these lectures are designed to balance investment portfolios, not trading portfolios.

Exercise. Compute μ and v based on daily data for the Markowitz portfolio with value equally distributed among the assets in each of the following groups:

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Exercise. The volatility of a portfolio is $\sigma = \sqrt{v}$. In the $\sigma\mu$-plane plot (σ, μ) for the two 2007 portfolios and (σ_i, m_i) for each of the 2007 assets in the previous exercise.

Exercise. In the $\sigma\mu$-plane plot (σ, μ) for the two 2009 portfolios and (σ_i, m_i) for each of the 2009 assets in the first exercise above.