Modeling Portfolios that Contain Risky Assets

Risk and Reward I: Introduction

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Risk and Reward I: Introduction

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Risk and Reward I: Introduction

Suppose you are considering how to invest in N risky assets that are traded on a market that had D trading days last year. (Typically $D = 255$.) Let $s_i(d)$ be the share price of the i^{th} asset at the close of the d^{th} trading day of the past year, where $s_i(0)$ is understood to be the share price at the close of the last trading day before the beginning of the past year. We will assume that every $s_i(d)$ is positive. You would like to use this price history to gain insight into how to manage your portfolio over the coming year.

We will examine the following questions.

Can stochastic (random, probabilistic) models be built that quantitatively mimic this price history? How can such models be used to help manage a portfolio?

Risky Assets. *The risk associated with an investment is the uncertainty of its outcome. Every investment has risk associated with it.* Hiding your cash under a mattress puts it at greater risk of loss to theft or fire than depositing it in a bank, and is a sure way to not make money. Depositing your cash into an FDIC insured bank account is the safest investment that you can make — the only risk of loss would be to an extreme national calamity. However, a bank account generally will yield a lower reward on your investment than any asset that has more risk associated with it. Such assets include stocks (equities), bonds, commodities (gold, oil, corn, etc.), private equity (venture capital), hedge funds, and real estate. With the exception of real estate, it is not uncommon for prices of these assets to fluctuate one to five percent in a day. Such assets are called *risky assets*.

Remark. *Market forces generally will insure that assets associated with higher potential reward are also associated with greater risk and vice versa.* Investment offers that seem to violate this principle are always scams.

Here we will consider three basic types of risky assets: *stocks*, *bonds*, and *commodities*. We will also consider *funds* that hold a combination of stocks, bonds, and/or commodities.

Stocks. Stocks are part ownership of a company. Their value goes up when the company does well, and goes down when it does poorly. Some stocks pay a periodic (usually quarterly) dividend of either cash or more stock. Stocks are traded on exchanges like the NYSE or NASDAQ.

The risk associated with a stock reflects the uncertainty about the future performance of the company. This uncertainty has many facets. For example, there might be questions about the future market share of its products, the availablity of the raw materials needed for its products, or the value of its current assets. Stocks in larger companies are generally less risky than stocks in smaller companies. *Stocks are generally higher reward/higher risk investments compared to bonds.*

Bonds. Bonds are loans to a government or company. The borrower usually makes a periodic (often quarterly) interest payment, and ultimately pays back the principle at a maturity date. Bonds are traded on secondary markets where their value is based on current interest rates. For example, if interest rates go up then bond values will go down on the secondary market.

The risk associated with a bond reflects the uncertainty about the credit worthiness of the borrower. Short term bonds are generally less risky than long term ones. Bonds from large entities are generally less risky than those from small entities. Bonds from governments are generally less risky than those from companies. (This is even true in some cases where the ratings given by some ratings agencies suggest otherwise.) *Bonds are generally lower reward/lower risk investments compared to stocks.*

Commodities. Commodities are hard assets such as gold, corn, oil, and real estate. They are bought in anticipation of their value going up. For example, this would be the case when an investor fears an inflationary period. Some commodities are bought in standard units (troy ounces, bushels, barrels). Others are bought through shares of a partnership. Some commodities like rental real estate will have regular income associated with it, but most provide no income.

The risk associated with a commodity reflects the uncertainty about the future demand for it and the future supply of it. This uncertainty has many facets because the variety of commodities is huge. For example, farm commodities are perishable, so will become worthless if they are held too long. The value of oil or gold will fall when new supplies are discovered. The demand for oil is generally higher during the northern hemisphere winter. Gold prices will spike during times of uncertainty, but tend to return to inflation adjusted levels. *Because of their variety, commodities can fall anywhere on the reward/risk spectrum.*

Mutual and Exchange-Traded Funds. These funds hold a combination of stocks, bonds, and/or commodities. Funds are set up by investment companies. Shares of mutual funds are bought and sold through the company that set it up. Shares of exchange-traded funds (ETFs) are bought and sold just as you would shares of a stock. *Funds are generally lower reward/lower risk investments compared to the individual assets from which they are composed.*

Funds are managed in one of two ways: *actively* or *passively*. An actively-managed fund usually has a strategy to perform better than some market index like the S&P 500, Russell 1000, or Russell 2000. A passively-managed fund usually builds a portfolio so that its performance will match some market index, in which case it is called an *index fund*. Index funds are often portrayed to be *lower reward/lower risk* investments compared to actively-managed funds. *However, index funds will typically outperform most actively-managed funds.* Reasons for this include the facts that they have lower management fees and that they require smaller cash reserves.

Return Rates. The first thing you must understand that the share price of an asset has very little economic significance. This is because the size of your investment in an asset is the same if you own 100 shares worth 50 dollars each or 25 shares worth 200 dollars each. What is economically significant is how much your investment rises or falls in value. Because your investment in asset i would have changed by the ratio $s_i(d)/s_i(d-1)$ over the course of day d, this ratio is economically significant. Rather than use this ratio as the basic variable, it is customary to use the so-called *return rate*, which we define by

$$r_i(d) = D\,\frac{s_i(d) - s_i(d-1)}{s_i(d-1)}\,.$$

The factor D arises because rates in banking, business, and finance are usually given as annual rates expressed in units of either "per annum" or "% per annum." Because a day is $\frac{1}{D}$ years the factor of D makes $r_i(d)$ a "per annum" rate. It would have to be multiplied by another factor of 100 to make it a "% per annum" rate. We will always work with "per annum" rates.

One way to understand return rates is to set $r_i(d)$ equal to a constant μ. Upon solving the resulting relation for $s_i(d)$ we find that

$$s_i(d) = \left(1 + \tfrac{\mu}{D}\right) s_i(d-1) \quad \text{for every } d = 1, \cdots, D\,.$$

By induction on d we can then derive the compound interest formula

$$s_i(d) = \left(1 + \tfrac{\mu}{D}\right)^d s_i(0) \quad \text{for every } d = 1, \cdots, D\,.$$

If we assume that $|\mu/D| << 1$ then we can see that

$$\left(1 + \tfrac{\mu}{D}\right)^{\frac{D}{\mu}} \approx \lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e\,,$$

whereby

$$s_i(d) = \left(1 + \tfrac{\mu}{D}\right)^{\frac{D}{\mu}\,\mu\,\frac{d}{D}} s_i(0) \approx e^{\mu\,\frac{d}{D}}\, s_i(0) = e^{\mu t} s_i(0)\,,$$

where $t = d/D$ is the time (in units of years) at which day d occurs. We thereby see μ is nearly the exponential growth rate of the share price.

We will consider a market of N risky assets indexed by i. For each i we obtain the closing share price history $\{s_i(d)\}_{d=0}^{D}$ of asset i over the past year, and compute the return rate history $\{r_i(d)\}_{d=1}^{D}$ of asset i over the past year by the formula

$$r_i(d) = D\,\frac{s_i(d) - s_i(d-1)}{s_i(d-1)}\,.$$

Because return rates are differences, we will need the closing share price from the day before the first day for which we want the return rate history.

You can obtain share price histories from websites like *Yahoo Finance* or *Google Finance*. For example, to compute the daily return rate history for Apple in 2010, type “Apple” into where is says “get quotes”. You will see that Apple has the identifier AAPL and is listed on the NASDAQ. Click on “historical prices” and request share prices between “Dec 31, 2009” and “Dec 31, 2010”. You will get a table that can be downloaded as a spreadsheet. The return rates are computed using the *closing prices*.

Remark. It is not obvious that return rates are the right quantities upon which to build a theory of markets. For example, another possibility is to use the *growth rates* $x_i(d)$ defined by

$$x_i(d) = D \, \log\left(\frac{s_i(d)}{s_i(d-1)}\right) .$$

These are also functions of the ratio $s_i(d)/s_i(d-1)$. Moreover, they seem to be easier to understand than return rates. For example, if we set $x_i(d)$ equal to a constant γ then by solving the resulting relation for $s_i(d)$ we find that

$$s_i(d) = e^{\frac{1}{D}\gamma} \, s_i(d-1) \quad \text{for every } d = 1, \cdots, D \, .$$

By induction on d we can then show that

$$s_i(d) = e^{\frac{d}{D}\gamma} s_i(0) \quad \text{for every } d = 1, \cdots, D \, ,$$

whereby $s_i(d) = e^{\gamma t} s_i(0)$ with $t = d/D$. *However, return rates have better properties with regard to porfolio statistics.*

Insights from Simple Models. We can use models to gain understanding as well as to reflect reality. Here we use a simple (unrealistic) stochastic (random) model of return rates to gain insights into aspects of markets.

Consider a market model in which each year every asset either goes up 20% or remains unchanged with equal probability. The mean return rate for this market is 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.2	1.2	1.44	1.2000
Second	1.2	1.0	1.20	1.0954
Third	1.0	1.2	1.20	1.0954
Fourth	1.0	1.0	1.00	1.0000

Three quarters of the investors see returns below 10%. One quarter of the investors have made no money.

The previous result did not look to bad for most investors, but perhaps you know that sometimes markets produce negative returns. Now consider a market model in which each year every asset either goes up 30% or goes down 10% with equal probability. The mean return rate for this market is also 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.3	1.3	1.69	1.3000
Second	1.3	0.9	1.17	1.0817
Third	0.9	1.3	1.17	1.0817
Fourth	0.9	0.9	0.81	0.9000

Three quarters of the investors see returns just above 8%. One quarter of the investors have lost money.

Next, consider a market model in which each year every asset either goes up 40% or goes down 20% with equal probability. The mean return rate for this market is also 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.4	1.4	1.96	1.4000
Second	1.4	0.8	1.12	1.0583
Third	0.8	1.3	1.12	1.0583
Fourth	0.8	0.8	0.64	0.8000

Three quarters of the investors see returns below 6%. One quarter of the investors have lost over one third of their investment.

Consider a market model in which each year every asset either goes up 50% or goes down 30% with equal probability. The mean return rate for this market is 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.5	1.5	2.25	1.5000
Second	1.5	0.7	1.05	1.0247
Third	0.7	1.5	1.05	1.0247
Fourth	0.7	0.7	0.49	0.7000

Three quarters of the investors see returns below 2.5%. One quarter of the investors have lost over half of their investment.

We see that fewer investers do well in more volatile markets. Therefore measures of volatility can be viewed a measures of risk.

There is another important insight to be gained from these models. Notice that if an individual investor buys an equal value of each asset in each of these markets then that investor sees a 10% return because

$$\frac{1.44 + 1.20 + 1.20 + 1.00}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2\,,$$
$$\frac{1.69 + 1.17 + 1.17 + 0.81}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2\,,$$
$$\frac{1.96 + 1.12 + 1.12 + 0.64}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2\,,$$
$$\frac{2.25 + 1.05 + 1.05 + 0.49}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2\,.$$

This shows the value of holding a diverse portfolio — specifically, that it reduces risk. We would like to understand how to apply this insight when designing real portfolios. This will require more realisitic models.

Statistical Approach. Return rates $r_i(d)$ for asset i can vary wildly from day to day as the share price $s_i(d)$ rises and falls. Sometimes the reasons for such fluctuations are very clear because they directly relate to some news about the company, agency, or government that issued the asset. For example, news of the Deepwater Horizon explosion caused the share price of British Petroleum stock to fall. At other times they relate to news that benefit or hurt entire sectors of assets. For example, a rise in crude oil prices might benefit oil and railroad companies but hurt airline and trucking companies. And at yet other times they relate to general technological, demographic, or social trends. For example, new internet technology might benefit Google and Amazon (companies that exist because of the internet) but hurt traditional “brick and mortar” retailers. Finally, there is often no evident public reason for a particular stock price to rise or fall. The reason might be a takeover attempt, a rumor, insider information, or the fact a large investor needs cash for some other purpose.

Given the complexity of the dynamics underlying such market fluctuations, we adopt a statistical approach to quantifying their trends and correlations. More specifically, we will choose an appropriate set of statistics that will be computed from selected return rate histories of the relevant assets. We will then use these statistics to calibrate a model that will predict how a set of ideal portfolios might behave in the future.

The implicit assumption of this approach is that in the future the market will behave statistically as it did in the past. This means that the data should be drawn from a long enough return rate history to sample most of the kinds of market events that we expect to see in the future. However, the history should not be too long because very old data will not be relevant to the current market. *To strike a balance we will use the return rate history from the most recent twelve month period, which we will dub "the past year".* For example, if we are planning our portfolio at the beginning of July 2011 then we will use the return rate histories for July 2010 through June 2011. Then D would be the number of trading days in this period.

Suppose that we have computed the return rate history $\{r_i(d)\}_{d=1}^{D}$ for each asset over the past year. At some point this data should be ported from the speadsheet into MATLAB, R, or another higher level environment that is well suited to the task ahead.

Mean-Variance Models. The next step is to compute the statistical quantities we will use in our models: *means, variances, covariances,* and *correlations*.

The return rate *mean* for asset i over the past year, denoted m_i, is

$$m_i = \frac{1}{D} \sum_{d=1}^{D} r_i(d) \, .$$

This measures the trend of the share price. Unfortunately, it is commonly called the *expected return rate* for asset i even though it is *higher* than the return rate that most investors will see, especially in highly volatile markets. We will not use this misleading terminology.

The return rate *variance* for asset i over the past year, denoted v_i, is

$$v_i = \frac{1}{D(D-1)} \sum_{d=1}^{D} \Big(r_i(d) - m_i\Big)^2 .$$

The reason for the $D(D-1)$ in the denominator will be made clear later. The return rate *standard deviation* for asset i over the year, denoted σ_i, is given by $\sigma_i = \sqrt{v_i}$. This is called the *volatility* of asset i. It measures the uncertainty of the market regarding the share price trend.

The *covariance* of the return rates for assets i and j over the past year, denoted v_{ij}, is

$$v_{ij} = \frac{1}{D(D-1)} \sum_{d=1}^{D} \Big(r_i(d) - m_i\Big)\Big(r_j(d) - m_j\Big) .$$

Notice that $v_{ii} = v_i$. The $N \times N$ matrix (v_{ij}) is symmetric and nonnegative definite. It will usually be positive definite — so we will assume it to be so.

Finally, the *correlation* of the return rates for assets i and j over the past year, denoted c_{ij}, is

$$c_{ij} = \frac{v_{ij}}{\sigma_i \, \sigma_j} \,.$$

Notice that $-1 \leq c_{ij} \leq 1$. We say assets i and j are *positively correlated* when $0 < c_{ij} \leq 1$ and *negatively correlated* when $-1 \leq c_{ij} < 0$. Positively correlated assets will tend to move in the same direction, while negatively correlated ones will often move in opposite directions.

We will consider the N-vector of means (m_i) and the symmetric $N \times N$ matrix of covariances (v_{ij}) to be our basic statistical quantities. We will build our models to be consistent with these statistics. The variances (v_i), volatilities (σ_i), and correlations (c_{ij}) can then be easily obtained from (m_i) and (v_{ij}) by formulas that are given above. The computation of the statistics (m_i) and (v_{ij}) from the return rate histories is called the *calibration* of our models.

Remark. Here the trading day is an arbitrary measure of time. From a theoretical viewpoint we could equally well have used a shorter measure like half-days, hours, quarter hours, or minutes. The shorter the measure, the more data has to be collected and analyzed. This extra work is not worth doing unless you profit sufficiently. Alternatively, we could have used a longer measure like weeks, months, or quarters. The longer the measure, the less data you use, which means you have less understanding of the market. For many investors daily or weekly data is a good balance. If you use weekly data $\{s_i(w)\}_{w=0}^{52}$, where $s_i(w)$ is the share price of asset i at the end of week w, then the rate of return of asset i for week w is

$$r_i(w) = 52\,\frac{s_i(w) - s_i(w-1)}{s_i(w-1)}\,.$$

You have to make consistent changes when computing m_i, v_i, and v_{ij} by replacing d with w and D with 52 in their defining formulas.

General Calibration. We can consider a history $\{r(d)\}_{d=1}^{D_h}$ over a period of D_h trading days and assign day d a weight $w(d) > 0$ such that the weights $\{w(d)\}_{d=1}^{D_h}$ satisfy

$$\sum_{d=1}^{D_h} w(d) = 1\,.$$

The return rate means and covariances are then given by

$$m_i = \sum_{d=1}^{D_h} w(d)\, r_i(d)\,,$$

$$v_{ij} = \frac{1}{D} \sum_{d=1}^{D_h} \frac{w(d)}{1 - \bar{w}} \Big(r_i(d) - m_i\Big)\Big(r_j(d) - m_j\Big)\,,$$

where

$$\bar{w} = \sum_{d=1}^{D_h} w(d)^2\,.$$

In practice the history will extend over a period of one to five years. There are many ways to choose the weights $\{w(d)\}_{d=1}^{D_h}$. The most common choice is the so-called *uniform weighting*; this gives each day the same weight by setting $w(d) = 1/D_h$. On the other hand, we might want to give more weight to more recent data. For example, we can give each trading day a positive weight that depends only on the quarter in which it lies, giving greater weight to more recent quarters. We could also consider giving different weights to different days of the week, but such a complication should be avoided unless it yields a clear benefit.

We will have greater confidence in m_i and v_{ij} when they are relatively insensitive to different choices of D_h and the weights $w(d)$. We can get an idea of the magnitude of this sensitivity by checking the robustness of m_i and v_{ij} to a range of such choices.

Exercise. Compute m_i, v_i, v_{ij}, and c_{ij} for each of the following groups of assets based on daily data, weekly data, and monthly data:

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Give explanations for the values of c_{ij} you computed.

Exercise. Compute m_i, v_i, v_{ij}, and c_{ij} for the assets listed in the previous exercise based on daily data and weekly data, but only from the last quarter of the year indicated. Based on a comparison of these answers with those of the previous problem, in which numbers might you have the most confidence, the m_i, v_i, v_{ij}, or c_{ij}?