Optimizing over Interventions within a Budget

Brian Hunt
University of Maryland
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Budget Constraint

- Recall our 2-group SI model with type a and b interventions:

$$
\begin{aligned}
dS_1/dt &= -p_{11}S_1\mathcal{I}_1 - p_{12}S_1\mathcal{I}_2 - a_1S_1 \\
d\mathcal{I}_1/dt &= p_{11}S_1\mathcal{I}_1 + p_{12}S_1\mathcal{I}_2 - (a_1 + b_1)\mathcal{I}_1 \\
dS_2/dt &= -p_{21}S_2\mathcal{I}_1 - p_{22}S_2\mathcal{I}_2 - a_2S_2 \\
d\mathcal{I}_2/dt &= p_{21}S_2\mathcal{I}_1 + p_{22}S_1\mathcal{I}_2 - (a_2 + b_2)\mathcal{I}_2.
\end{aligned}
$$

- The impact $M(a_1, a_2, b_1, b_2)$ (fraction of the inital at-risk population saved from infection by the intervention) is an increasing function of each parameter a_1, a_2, b_1, b_2.
- We now consider the problem of maximizing M subject to a "budget" constraint $K(a_1, a_2, b_1, b_2) \leq K_{\text{max}}$ where $K(a_1, a_2, b_1, b_2)$ is the cost of achieving parameters a_1, a_2, b_1, b_2.

Linear Budget Functions

- The simplest class of cost functions are linear functions $K(a_1, a_2, b_1, b_2) = c_{a1}a_1 + c_{a2}a_2 + c_{b1}b_1 + c_{b2}b_2$ where the c's are positive constants representing "marginal" costs.
- To simplify further, let's assume $c_{a1} = c_{a2}$ and $c_{b1} = c_{b2}$ and let $c = c_{a1}/c_{b1} = c_{a2}/c_{b2}$.
- Let's normalize (choose units of cost) so that $c_{b1} = c_{b2} = 1$, and call the resulting cost function K_c.
- The main flaw in a linear cost function is that it doesn't have the "diminishing returns" observed in real life.

Constrained Optimization

- We are considering a constrained optimization problem; in addition to the constraint $K_c(a_1, a_2, b_1, b_2) \leq K_{\max}$ we have $a_1, a_2, b_1, b_2 \geq 0$. These inequalities describe a 4-dimensional simplex over which we want to maximize the impact M.
- Since M is an increasing function of the parameters a_1, a_2, b_1, and b_2, the maximum impact will occur when the entire budget is used: $K_c(a_1, a_2, b_1, b_2) = K_{\max}$. This equality allows one parameter to be determined from the other three, reducing the domain to a 3-dimensional simplex – a tetrahedron.
- The maximum of M often occurs when one or more of the parameters is zero, meaning that the maximum occurs on one of the faces, edges, or vertices of the tetrahedron.

Optimization Strategy

- Iterative, “guess-and-perturb” optimization algorithms are problematic for constrained optimization because the allowed perturbations depend on whether one is inside the constraint domain or on its boundary, and where on the boundary.
- A simpler approach, feasible with a few parameters, is to search the entire domain to a certain resolution – choose a closely-spaced grid and search over the grid points in the domain.
- To distinguish between the boundary and the interior of the domain, you need to sample all parts of the boundary. For the tetrahedron described on the previous slide, a rectangular grid will sample the faces and edges that are aligned with the coordinate axes, but you may need to choose additional points to sample the diagonal face and edges.