Modified SIR Model and Nondimensionalization

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Modified SIR Model

- A standard extension to the SIR model adds terms representing births and deaths that are proportional to the overall population.
- If we're modeling an adult subpopulation that is either infected or "at risk", it may be more appropriate to add a net influx that is independent of the current susceptible/infectious/removed populations:

$$dS/dt = q - pS\mathcal{I}$$
$$d\mathcal{I}/dt = pS\mathcal{I} - r\mathcal{I}$$
$$dR/dt = r\mathcal{I}$$

- The cumulative number of people infected is $\mathcal{I} + R$, and the rate of new infections is $pS\mathcal{I}$.

Units of Variables and Parameters

- The variables S, $\mathcal{I}$ and R have units of "population". A unit of population could be one person, but sometimes other units are used; e.g., census data is often tabulated in units of thousands of people.
- The derivatives dS/dt, $d\mathcal{I}/dt$, dR/dt have units of population/time. Therefore:
  - p has units of 1/(population $\cdot$ time).
  - q has units of population/time.
  - r has units of 1/time.

Change of Parameters

- In the SI model, we developed new parameters N, λ, and δ that had units of population, 1/time, and time, respectively.
- This separated a parameter N that controlled the size of the outbreak from parameters λ and δ that controlled the speed and timing of the outbreak.
- Let's do something similar for our modified SIR model.
- If we let $N = S + \mathcal{I} + R$, then N changes over time (if $q \neq 0$). To make N a constant, let's set $N = S(0) + \mathcal{I}(0) + R(0)$.

Normalized/Nondimensional Parameters

- Let's make N be the only parameter whose units involve population.
- As before, let $\lambda = pN$; then λ has units of 1/time.
- Let's make the other parameters be "dimensionless" – i.e., be independent of the units of time and population.
- Let $\mu = q/(\lambda N)$; then μ is dimensionless.
- Let $\nu = r/\lambda$; then ν is dimensionless.

Model with New Parameters

- The modified SIR model with the new parameters is

$$
\begin{aligned}
dS/dt &= \lambda(\mu N - S\mathcal{I}/N) \\
d\mathcal{I}/dt &= \lambda(S\mathcal{I}/N - \nu\mathcal{I}) \\
dR/dt &= \lambda\nu\mathcal{I}
\end{aligned}
$$

- This is a bit messier than before, but it allows us to isolate better the effects of changing a parameter.
- If we multiply N by a constant a, then multiplying S, $\mathcal{I}$,R by the same constant yields a solution to the model with the same values of λ,μ,ν.
- If we change λ, this only changes the rate of the outbreak (not the size).

Interpretation of New Parameters

- N controls the size of the outbreak. However, it does not represent the total number who will eventually be infected.
- λ controls the rate of the outbreak.
- μ and ν are rates of the renewal (new susceptibles) and recovery/removal processes relative to the infection process.
- There are two more parameters that determine the relative sizes of $S(0)$, $\mathcal{I}(0)$, and $R(0)$, whose sum we have called N. For example, we could use dimensionless parameters $\alpha = \mathcal{I}(0)/N$ and $\beta = R(0)/N$. If time 0 is early in the outbreak, it may be reasonable to assume that β is essentially 0.

Questions to Consider

- What happens to $S(t)$, $\mathcal{I}(t)$, $R(t)$, and to the rate of new infections as $t \to \infty$? The most relevant parmaters are μ and ν, which determine the “shape” of the solutions, as opposed to N and λ, which determine their size and speed. The relative sizes of the initial conditions may also be relevant.
- How much do the new processes (renewal, governed by q and μ, and recovery/removal, governed by r and ν) affect the model’s ability to fit a data set? (It may help to consider separately the cases $q = \mu = 0$ and $r = \nu = 0$.)